Preface

This book is for a one-semester undergraduate real analysis course, taught here at M.I.T. for about 25 years, and in its present form for about 15. It runs in parallel with a more difficult course based on point-set topology.

Its origins Many years ago, my older brother Dick, then a very smart graduate student in theoretical physics here, came in fuming.

"I'm taking this graduate complex variable course, and I can't understand a word of it. The book (it was Ahlfors) just goes on and on in English about open sets and closed sets, and then says 'This concludes the proof.' I was always great at mathematics. What am I doing wrong?"

As he was my chief tormenter in childhood, my advice was a smug, "Just keep on trying, I'm sure you'll get it."

A month later he came in again.

"Last night I suddenly saw it, and it's all completely trivial."

Years later, a senior physics undergraduate sat down before me and sighed.

"Well, this is the fourth time I'm dropping analysis. Each time I get a little further into the course, but the open sets always win out in the end. Isn't it possible to teach it so guys like me could understand it? We understand derivations, but they give us proofs instead. Inequalities are OK, as long as they look like equations, but this analysis doesn't look like the math we know — it's all in English instead of symbols. And as far as any of us can tell, the only thing any theorem is good for is proving the next theorem."

Kenneth Hoffman agreed to see what could be done; then Frank Morgan and Steven Robbins each taught it for a year. Afterwards, profiting from their experiences and the notes they generously left me, I took it over and developed the notes from which this book has evolved.

The course The course is basically one-variable analysis. The emphasis throughout is not on the algebraic or topological aspects of analysis, but on estimation and approximation: how analysis replaces the equalities of calculus with inequalities: certainty with uncertainty. This represents for students a step up in maturity.

To help, arguments use as little English as possible, and are formulated to look like successions of equations or inequalities: derivations, in other words.

Calculus is used freely from the beginning as a source of examples, so students can see how the ideas are used. The real numbers are discussed briefly in the first chapter, with most of the emphasis on the completeness property. The aim is to get to interesting things as quickly as possible. Several appendices present extended applications.

Needless to say, point-set topology, the *pons asinorum* of analysis courses, has been banished to near the end, and presented in abbreviated form, just before it is needed in the study of integrals depending on a parameter. By then, students can understand the arguments, and even enjoy them as something new-looking.

A few times I got through the whole book, but nowadays don't seem to get that far. There are different goals one can aim at, any of which provide a sense of fulfillment, since they justify results known from previous courses but unproved there, or generally provoke curiosity:

⋆ differentiating the Laplace transform under the integral sign (Section 27.5);

⋆ the existence and uniqueness theorem for first order ODE's (Appendix E);

⋆ differentiating a power series term-by-term (Chapter 22);

⋆ what the Lebesgue integral does for you (Chapter 23);

or just the theorems of calculus revisited (Chapter 20 or 21), with an easy chapter illustrating some new uses for them (Appendix C). For each of these goals, certain chapters can be skipped, which teachers will be able to identify without trouble.

Features of the book Some account of what I take to be distinguishing features of the book may be helpful.

Questions At the end of each short section (1-3 pages) are questions, with answers at the end of the chapter. (At the end of the chapter are the customary exercises, tied to the sections, and also somewhat harder problems, going with the chapter as a whole.) The questions have many functions.

(a) There are many worked-out examples in the book; if there were more, the main thread of the exposition would tend to get lost. But students always want more examples. The questions supply them. And because they have to turn over some pages to get the answers, there's at least some hope they will actually try them without first peeking. It's an unusual student who will do this with worked-out examples in the text.

(b) Answers are often complete proofs, written in proper style, that students can use as models.

(c) The questions act as traffic signals, not-so-gently prodding the students to stop and see if they understand what they have just read, instead of mindlessly continuing with their yellow highlighting. Some questions ask about specific things in the proofs. Occasionally a proof will deliberately gloss over some point, leaving the more detailed explanation as the answer to a question.

Writing Most of the students taking this course have not seriously studied proofs before, and haven't had to write any of their own. A major goal of the book is to get them to be able to do this. So there are remarks in the first few chapters (towards the ends of Chapters 3 and 5 for instance) about how to write up arguments, as well as warnings sprinkled here and there about common pitfalls to avoid. Some of these are in the answers to questions.

A lot of this was written in the wan hope of getting assignments that were easier to read and grade, I must confess.

Reading and Typography: a Diatribe The ability to write proofs goes hand-in-hand with the ability to read them. To facilitate this I have tried hard to make the book readable, for example by asking average students to note in the margins everything that puzzled them on a first reading. (I learned a lot about the placement of subordinate clauses from this, and recommend it heartily to fellow-sufferers in scrivening.)

In my opinion, many otherwise good books are spoiled by indifference to layout. How can one expect students to write arguments decently, when they study from books where a proof is presented as a solid block of type, equations broken in the middle, the final line and crux of a proof appearing all by itself on the other side of the page? Decent typesetters didn't do these things; alas, some TeX-happy authors do.

But TeX can be a blessing instead of a curse: it allows arguments and formulas to be set out in the clearest format; one can experiment on the spot, rewriting sentences to get a better layout, something the best compositor cannot do.

I've tried to avail myself of these possibilities: almost no proofs require a page turn; implication arrows are lined up for maximum clarity, spacing in formulas, mathematical phrases, and between paragraphs has been adjusted to correspond to the pauses one would make in reading aloud. I hope there are no frightening pages here.

Dept. of Fuller Explanation To make analysis more accessible, the book goes into more detail than usual about elementary things concerning functions, inequalities, and so on.

About all I can say in defense of this is that these are things which I find that many of my students don't seem to know, or don't know explicitly. They subtract inequalities, are vague about inverse functions, and not always sure just what functions are or how they should think about them.

Mathematical notation The book makes use of some non-traditional notation and terminology, which classes have received well, while (more to the point) their teachers have not objected too strenuously.

Proofs are allowed to end with 42ϵ instead of ϵ — in the book, this bears the name "K-ϵ principle", introduced explicitly in Chapter 3, and used throughout.

When the approximation is the main idea to be expressed, the estimate $|a - b| < \epsilon$ is often written $a \underset{\epsilon}{\approx} b$. This too is harmless.

More serious is the use of these terms borrowed from applied mathematics: "for $n \gg 1$" (for large n) and "for $x \approx a$" (for x sufficiently close to a).

The former for example is introduced at the end of Chapter 2 and used right away at the beginning of Chapter 3 in the definition of the limit of a sequence. By avoiding the explicit use of an N or δ, these terms provide a very gentle introduction to limits, and suppress a lot of unnecessary details in later arguments. I've used them with classes for thirty years, and would not go back to using N and δ routinely. (When needed for an argument, they can be introduced by a phrase such as "for $n \gg 1$, say for $n > N$".)

The main places where N or δ must be made explicit are:

⋆ the very beginning, in proving that some expression really has a given number L as its limit;

⋆ in the proof of limit theorems involving composite functions (there are only two or three; sequential continuity is one);

⋆ in the definition of uniform continuity and uniform convergence; these occur later in the course, however, and students can by then handle the complexity of another quantifier;

⋆ in forming the negations needed for negative arguments.

This last is serious. The average student cannot negate "$a_n > 0$ for all n" correctly. If you teach students how to do it (see Appendix B), they start making even the simplest argument negative, a terrible habit. Therefore, the book avoids negative arguments when possible, and handles the negations informally when they become necessary (principally in Chapter 13).

Appendix A discusses negative arguments, but warns against using them as first choice. Appendix B is the forbidden fruit — it discusses both quantifiers and negation explicitly. Classes that read it early are I think asking for trouble.

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