18.100C Lecture 19 Summary

A partition P of $[a, b]$ is given by $a = x_0 < x_1 < \cdots < x_n = b$, for some n.

Definition 19.1. *A function $f : [a, b] \to \mathbb{R}$ is called* piecewise linear *if there is a partition P such that f is linear (of the form $c_i x + d_i$) on each interval $[x_{i-1}, x_i]$.*

To get integration theory started, one defines the integral of a piecewise linear function to be

$$\int_a^b f(x)\, dx \stackrel{\text{def}}{=} \sum_{i=1}^{n} \tfrac{1}{2}(f(x_{i-1}) + f(x_i))\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$. This is the geometric formula for the area of a trapezoid (with sign), added up over i. Again by elementary geometric arguments, the integral of a piecewise linear function is independent of the choice of partition.

Proposition 19.2. *(i) If $f(x) = c$ is constant, $\int_a^b f(x)dx = c(b - a)$.*

(ii) Suppose that f, g are piecewise linear functions. Then

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

(iii) Suppose that f is a piecewise linear function and c is a constant. Then

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx.$$

(iv) If a piecewise linear function f satisfies $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\, dx \geq 0$.

We extend this notion of integral to all continuous functions by uniform convergence.

Lemma 19.3. *Any continuous function $f : [a, b] \to \mathbb{R}$ is the uniform limit of a sequence of piecewise linear functions.*

Theorem 19.4. *There is a unique way of assigning to each continuous $f : [a, b] \to \mathbb{R}$ a number $\int_a^b f(x)\, dx \in \mathbb{R}$ such that: if f is piecewise linear, then we get back the integral as defined before; and if $f_n \to f$ uniformly, then $\int_a^b f_n(x)\, dx \to \int_a^b f(x)\, dx$.*

The integral of continuous functions defined in this way has the same properties as that for piecewise linear functions stated in the Proposition above.

Definition 19.5. *A function $f : [a, b] \to \mathbb{R}$ is a step function if there is a partition P such that f is constant on each open interval (x_{i-1}, x_i).*

One can use the geometric formula for the integral of a step function (sum of areas of rectangles) as a starting point, and then extend that by uniform convergence. This yields a notion of integral which covers all continuous functions as well as some others (but still not as much as the Riemann integral).

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