18.102 Introduction to Functional Analysis
Spring 2009

PROBLEM SET 1 FOR 18.102, SPRING 2009
DUE 11AM TUESDAY 10 FEB.

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Full marks will be given to anyone who makes a good faith attempt to answer each question. The first four problems concern the 'little L p' spaces l^p. Note that you have the choice of doing everything for $p = 2$ or for all $1 \le p < \infty$.

1. Problem 1.1

Write out a proof (you can steal it from one of many places but at least write it out in your own hand) either for $p = 2$ or for each p with $1 \le p < \infty$ that

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold.

2. Problem 1.2

The 'tricky' part in Problem 1.1 is the triangle inequality. Suppose you knew – meaning I tell you – that for each N

$$\left(\sum_{j=1}^{N} |a_j|^p\right)^{\frac{1}{p}} \text{ is a norm on } \mathbb{C}^N$$

would that help?

3. Problem 1.3

Prove directly that each l^p as defined in Problem 1.1 – or just l^2 – is complete, i.e. it is a Banach space. At the risk of offending some, let me say that this means showing that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. Show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements at point N is Cauchy with respect to the norm in Problem 1.2 on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense (if you are doing $p = 2$ it is already the usual sense) and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it works.

4. Problem 1.4

Consider the 'unit sphere' in l^p – where if you want you can set $p = 2$. This is the set of vectors of length 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) Show that S is closed.
(2) Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e.g. by checking in Rudin).
(3) Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!

5. Problem 1.5

Show that the norm on any normed space is continuous.

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