18.102 Introduction to Functional Analysis
Spring 2009

P9.1: Periodic functions

Let $\mathbb{S}$ be the circle of radius 1 in the complex plane, centered at the origin, $\mathbb{S} = \{z; |z| = 1\}$.

(1) Show that there is a 1-1 correspondence

$$\mathcal{C}^0(\mathbb{S}) = \{u : \mathbb{S} \longrightarrow \mathbb{C},\ \text{continuous}\} \longrightarrow \\ \{u : \mathbb{R} \longrightarrow \mathbb{C};\ \text{continuous and satisfying } u(x + 2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}. \tag{21.40}$$

Solution: The map $E : \mathbb{R} \ni \theta \longmapsto e^{2\pi i\theta} \in \mathbb{S}$ is continuous, surjective and 2π-periodic and the inverse image of any point of the circle is precisly of the form $\theta + 2\pi\mathbb{Z}$ for some $\theta \in \mathbb{R}$. Thus composition defines a map

$$E^* : \mathcal{C}^0(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{R}),\ E^*f = f \circ E. \tag{21.41}$$

This map is a linear bijection.

(2) Show that there is a 1-1 correspondence

$$L^2(0, 2\pi) \longleftrightarrow \{u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}); u\big|_{(0,2\pi)} \in \mathcal{L}^2(0, 2\pi) \\ \text{and } u(x + 2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}/\mathcal{N}_P \tag{21.42}$$

where $\mathcal{N}_P$ is the space of null functions on $\mathbb{R}$ satisfying $u(x + 2\pi) = u(x)$ for all $x \in \mathbb{R}$.

Solution: Our original definition of $L^2(0, 2\pi)$ is as functions on $\mathbb{R}$ which are square-integrable and vanish outside $(0, 2\pi)$. Given such a function u we can define an element of the right side of (21.42) by assigning a value at 0 and then extending by periodicity

$$\tilde{u}(x) = u(x - 2n\pi),\ n \in \mathbb{Z} \tag{21.43}$$

where for each $x \in \mathbb{R}$ there is a unique integer n so that $x - 2n\pi \in [0, 2\pi)$. Null functions are mapped to null functions his way and changing the choice of value at 0 changes $\tilde{u}$ by a null function. This gives a map as in (21.42) and restriction to $(0, 2\pi)$ is a 2-sided invese.

(3) If we denote by $L^2(\mathbb{S})$ the space on the left in (21.42) show that there is a dense inclusion

$$\mathcal{C}^0(\mathbb{S}) \longrightarrow L^2(\mathbb{S}). \tag{21.44}$$

Solution: Combining the first map and the inverse of the second gives an inclusion. We know that continuous functions vanishing near the end-points of $(0, 2\pi)$ are dense in $L^2(0, 2\pi)$ so density follows.

So, the idea is that we can freely think of functions on $\mathbb{S}$ as 2π-periodic functions on $\mathbb{R}$ and conversely.

P9.2: Schrödinger's operator

Since that is what it is, or at least it is an example thereof:

$$-\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x),\ x \in \mathbb{R}, \tag{21.45}$$

(1) First we will consider the special case $V = 1$. Why not $V = 0$? – Don't try to answer this until the end!

Solution: The reason we take $V = 1$, or at least some other positive constant is that there is 1-d space of periodic solutions to $d^2u/dx^2 = 0$, namely the constants.

(2) Recall how to solve the differential equation

$$-\frac{d^2u(x)}{dx^2} + u(x) = f(x),\ x \in \mathbb{R}, \tag{21.46}$$

where $f(x) \in \mathcal{C}^0(\mathbb{S})$ is a continuous, 2π-periodic function on the line. Show that there is a unique 2π-periodic and twice continuously differentiable function, u, on $\mathbb{R}$ satisfying (21.46) and that this solution can be written in the form

$$u(x) = (Sf)(x) = \int_{0,2\pi} A(x,y)f(y) \tag{21.47}$$

where $A(x,y) \in \mathcal{C}^0(\mathbb{R}^2)$ satisfies $A(x+2\pi, y+2\pi) = A(x,y)$ for all $(x,y) \in \mathbb{R}$.

Extended hint: In case you managed to avoid a course on differential equations! First try to find a solution, igonoring the periodicity issue. To do so one can (for example, there are other ways) factorize the differential operator involved, checking that

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(\frac{dv}{dx} + v) \text{ if } v = \frac{du}{dx} - u \tag{21.48}$$

since the cross terms cancel. Then recall the idea of integrating factors to see that

$$\begin{aligned} \frac{du}{dx} - u &= e^x\frac{d\phi}{dx},\ \phi = e^{-x}u, \\ \frac{dv}{dx} + v &= e^{-x}\frac{d\psi}{dx},\ \psi = e^x v. \end{aligned} \tag{21.49}$$

Now, solve the problem by integrating twice from the origin (say) and hence get a solution to the differential equation (21.46). Write this out explicitly as a double integral, and then change the order of integration to write the solution as

$$u'(x) = \int_{0,2\pi} A'(x,y)f(y)dy \tag{21.50}$$

where A' is continuous on $\mathbb{R} \times [0, 2\pi]$. Compute the difference $u'(2\pi) - u'(0)$ and $\frac{du'}{dx}(2\pi) - \frac{du'}{dx}(0)$ as integrals involving f. Now, add to u' as solution to the *homogeneous* equation, for $f = 0$, namely $c_1e^x + c_2e^{-x}$, so that the new solution to (21.46) satisfies $u(2\pi) = u(0)$ and $\frac{du}{dx}(2\pi) = \frac{du}{dx}(0)$. Now, check that u is given by an integral of the form (21.47) with A as stated.

Solution: Integrating once we find that if $v = \frac{du}{dx} - u$ then

$$v(x) = -e^{-x}\int_0^x e^s f(s)ds,\ u'(x) = e^x\int_0^x e^{-t}v(t)dt \tag{21.51}$$

gives a solution of the equation $-\frac{d^2u'}{dx^2}+u'(x)=f(x)$ so combinging these two and changing the order of integration

(21.52)
$$u'(x)=\int_0^x \tilde A(x,y)f(y)dy,\ \tilde A(x,y)=\frac12\left(e^{y-x}-e^{x-y}\right)$$
$$u'(x)=\int_{(0,2\pi)}A'(x,y)f(y)dy,\ A'(x,y)=\begin{cases}\frac12\left(e^{y-x}-e^{x-y}\right) & x\ge y\\ 0 & x\le y.\end{cases}$$

Here A' is continuous since $\tilde A$ vanishes at $x=y$ where there might otherwise be a discontinuity. This is the only solution which vanishes with its derivative at 0. If it is to extend to be periodic we need to add a solution of the homogeneous equation and arrange that

(21.53)
$$u=u'+u'',\ u''=ce^x+de^{-x},\ u(0)=u(2\pi),\ \frac{du}{dx}(0)=\frac{du}{dx}(2\pi).$$

So, computing away we see that

(21.54)
$$u'(2\pi)=\int_0^{2\pi}\frac12\left(e^{y-2\pi}-e^{2\pi-y}\right)f(y),\ \frac{du'}{dx}(2\pi)=-\int_0^{2\pi}\frac12\left(e^{y-2\pi}+e^{2\pi-y}\right)f(y).$$

Thus there is a unique solution to (21.53) which must satify

(21.55)
$$c(e^{2\pi}-1)+d(e^{-2\pi}-1)=-u'(2\pi),\ c(e^{2\pi}-1)-d(e^{-2\pi}-1)=-\frac{du'}{dx}(2\pi)$$
$$(e^{2\pi}-1)c=\frac12\int_0^{2\pi}\left(e^{2\pi-y}\right)f(y),\ (e^{-2\pi}-1)d=-\frac12\int_0^{2\pi}\left(e^{y-2\pi}\right)f(y).$$

Putting this together we get the solution in the desired form:

(21.56)
$$u(x)=\int_{(0.2\pi)}A(x,y)f(y),\ A(x,y)=A'(x,y)+\frac12\frac{e^{2\pi-y+x}}{e^{2\pi}-1}-\frac12\frac{e^{-2\pi+y-x}}{e^{-2\pi}-1}\Longrightarrow$$
$$A(x,y)=\frac{\cosh(|x-y|-\pi)}{e^{\pi}-e^{-\pi}}.$$

(3) Check, either directly or indirectly, that $A(y,x)=A(x,y)$ and that A is real.

Solution: Clear from (21.56).

(4) Conclude that the operator S extends by continuity to a bounded operator on $L^2(\mathbb{S})$.

Solution. We know that $\|S\|\le\sqrt{2\pi}\sup|A|$.

(5) Check, probably indirectly rather than directly, that

(21.57)
$$S(e^{ikx})=(k^2+1)^{-1}e^{ikx},\ k\in\mathbb{Z}.$$

Solution. We know that Sf is the unique solution with periodic boundary conditions and e^{ikx} satisfies the boundary conditions and the equation with $f=(k^2+1)e^{ikx}$.

(6) Conclude, either from the previous result or otherwise that S is a compact self-adjoint operator on $L^2(\mathbb{S})$.

Soluion: Self-adjointness and compactness follows from (21.57) since we know that the $e^{ikx}/\sqrt{2\pi}$ form an orthonormal basis, so the eigenvalues of S

tend to 0. (Myabe better to say it is approximable by finite rank operators by truncating the sum).

(7) Show that if $g \in \mathcal{C}^0(\mathbb{S}))$ then Sg is twice continuously differentiable. Hint: Proceed directly by differentiating the integral.

Solution: Clearly Sf is continuous. Going back to the formula in terms of $u' + u''$ we see that both terms are twice continuously differentiable.

(8) From (21.57) conclude that $S = F^2$ where F is also a compact self-adjoint operator on $L^2(\mathbb{S})$ with eigenvalues $(k^2+1)^{-\frac{1}{2}}$.

Solution: Define $F(e^{ikx}) = (k^2+1)^{-\frac{1}{2}}e^{ikx}$. Same argument as above applies to show this is compact and self-adjoint.

(9) Show that $F : L^2(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{S})$.

Solution. The series for Sf

$$Sf(x) = \frac{1}{2\pi}\sum_k (2k^2+1)^{-\frac{1}{2}}(f, e^{ikx})e^{ikx} \tag{21.58}$$

converges absolutely and uniformly, using Cauchy's inequality – for instance it is Cauchy in the supremum norm:

$$\| \sum_{|k|>p} (2k^2+1)^{-\frac{1}{2}}(f, e^{ikx})e^{ikx}| \le \epsilon\|f\|_{L^2} \tag{21.59}$$

for p large since the sum of the squares of the eigenvalues is finite.

(10) Now, going back to the real equation (21.45), we assume that V is continuous, real-valued and 2π-periodic. Show that if u is a twice-differentiable 2π-periodic function satisfying (21.45) for a given $f \in \mathcal{C}^0(\mathbb{S})$ then

$$u + S((V-1)u) = Sf \text{ and hence } u = -F^2((V-1)u) + F^2 f \tag{21.60}$$

and hence conclude that

$$u = Fv \text{ where } v \in L^2(\mathbb{S}) \text{ satisfies } v + (F(V-1)F)v = Ff \tag{21.61}$$

where $V-1$ is the operator defined by multiplication by $V-1$.

Solution: If u satisfies (21.45) then

$$-\frac{d^2u(x)}{dx^2} + u(x) = -(V(x)-1)u(x) + f(x) \tag{21.62}$$

so by the uniquenss of the solution with periodic boundary conditions, $u = -S(V-1)u + Sf$ so $u = F(-F(V-1)u + Ff)$. Thus indeed $u = Fv$ with $v = -F(V-1)u + Ff$ which means that v satisfies

$$v + F(V-1)Fv = Ff. \tag{21.63}$$

(11) Show the converse, that if $v \in L^2(\mathbb{S})$ satisfies

$$v + (F(V-1)F)v = Ff,\ f \in \mathcal{C}^0(\mathbb{S}) \tag{21.64}$$

then $u = Fv$ is 2π-periodic and twice-differentiable on $\mathbb{R}$ and satisfies (21.45).

Solution. If $v \in L^2(0, 2\pi)$ satisfies (21.64) then $u = Fv \in \mathcal{C}^0(\mathbb{S})$ satisfies $u + F^2(V-1)u = F^2f$ and since $F^2 = S$ maps $\mathcal{C}^0(\mathbb{S})$ into twice continuously differentiable functions it follows that u satisfies (21.45).

(12) Apply the Spectral theorem to $F(V-1)F$ (including why it applies) and show that there is a sequence λ_j in $\mathbb{R}\setminus\{0\}$ with $|\lambda_j|\to 0$ such that for all $\lambda\in\mathbb{C}\setminus\{0\}$, the equation

$$\lambda v + (F(V-1)F)v = g,\ g\in L^2(\mathbb{S}) \tag{21.65}$$

has a unique solution for every $g\in L^2(\mathbb{S})$ if and only if $\lambda\neq\lambda_j$ for any j.

Solution: We know that $F(V-1)F$ is self-adjoint and compact so $L^2(0.2\pi)$ has an orthonormal basis of eigenfunctions of $-F(V-1)F$ with eigenvalues λ_j. This sequence tends to zero and (21.65), for given $\lambda\in\mathbb{C}\setminus\{0\}$, if and only if has a solution if and only if it is an isomorphism, meaning $\lambda\neq\lambda_j$ is not an eigenvalue of $-F(V-1)F$.

(13) Show that for the λ_j the solutions of

$$\lambda_j v + (F(V-1)F)v = 0, v\in L^2(\mathbb{S}), \tag{21.66}$$

are all continuous 2π-periodic functions on $\mathbb{R}$.

Solution: If v satisfies (21.66) with $\lambda_j\neq 0$ then $v=-F(V-1)F/\lambda_j\in\mathcal{C}^0(\mathbb{S})$.

(14) Show that the corresponding functions $u=Fv$ where v satisfies (21.66) are all twice continuously differentiable, 2π-periodic functions on $\mathbb{R}$ satisfying

$$-\frac{d^2u}{dx^2}+(1-s_j+s_jV(x))u(x)=0,\ s_j=1/\lambda_j. \tag{21.67}$$

Solution: Then $u=Fv$ satisfies $u=-S(V-1)u/\lambda_j$ so is twice continuously differentiable and satisfies (21.67).

(15) Conversely, show that if u is a twice continuously differentiable, 2π-periodic function satisfying

$$-\frac{d^2u}{dx^2}+(1-s+sV(x))u(x)=0,\ s\in\mathbb{C}, \tag{21.68}$$

and u is not identically 0 then $s=s_j$ for some j.

Solution: From the uniquess of periodic solutions $u=-S(V-1)u/\lambda_j$ as before.

(16) Finally, conclude that Fredholm's alternative holds for the equation (21.45)

Theorem 16. *For a given real-valued, continuous 2π-periodic function V on $\mathbb{R}$, either* (21.45) *has a unique twice continuously differentiable, 2π-periodic, solution for each f which is continuous and 2π-periodic or else there exists a finite, but positive, dimensional space of twice continuously differentiable 2π-periodic solutions to the homogeneous equation*

$$-\frac{d^2w(x)}{dx^2}+V(x)w(x)=0,\ x\in\mathbb{R}, \tag{21.69}$$

and (21.45) *has a solution if and only if $\int_{(0,2\pi)} fw=0$ for every 2π-periodic solution, w, to* (21.69).

Solution: This corresponds to the special case $\lambda_j=1$ above. If λ_j is not an eigenvalue of $-F(V-1)F$ then

$$v+F(V-1)Fv=Ff \tag{21.70}$$

has a unque solution for all f, otherwise the necessary and sufficient condition is that $(v,Ff)=0$ for all v' satisfying $v'+F(V-1)Fv'=0$. Correspondingly either

(21.45) has a unique solution for all f or the necessary and sufficient condition is that $(Fv', f) = 0$ for all $w = Fv'$ (remember that F is injetive) satisfying (21.69).

Not to be handed in, just for the enthusiastic

Check that we really can understand all the 2π periodic eigenfunctions of the Schrödinger operator using the discussion above. First of all, there was nothing sacred about the addition of 1 to $-d^2/dx^2$, we could add any positive number and get a similar result – the problem with 0 is that the constants satisfy the homogeneous equation $d^2u/dx^2 = 0$. What we have shown is that the operator

$$u \longmapsto Qu = -\frac{d^2u}{dx^2}u + Vu \tag{21.71}$$

applied to twice continuously differentiable functions has at least a left inverse unless there is a non-trivial solution of

$$-\frac{d^2u}{dx^2}u + Vu = 0. \tag{21.72}$$

Namely, the left inverse is $R = F(\mathrm{Id} + F(V-1)F)^{-1}F$. This is a compact self-adjoint operator. Show – and there is still a bit of work to do – that (twice continuously differentiable) eigenfunctions of Q, meaning solutions of $Qu = \tau u$ are precisely the non-trivial solutions of $Ru = \tau^{-1}u$.

What to do in case (21.72) does have a non-trivial solution? Show that the space of these is finite dimensional and conclude that essentially the same result holds by working on the orthocomplement in $L^2(\mathbb{S})$.