18.102 Introduction to Functional Analysis
Spring 2009

Lecture 4. Thursday, 12 Feb

I talked about step functions, then the covering lemmas which are the basis of the definition of Lebesgue measure – which we will do after the integral – then properties of monotone sequences of step functions.

To be definite, but brief, by an interval we will mean $[a, b)$ – an interval closed on the left and open on the right – at least for a little while. This is just so the length of the interval, $b - a$, only vanishes when the interval is empty (not true for closed intervals of course) and also so that we can decompose an interval, in this sense, into two disjoint intervals by choosing any interior point:

$$[a, b) = [a, t) \cup [t, b),\ a < t < b. \tag{4.1}$$

Now, by a step function

$$f : \mathbb{R} \longrightarrow \mathbb{C} \tag{4.2}$$

(although often we will restrict to functions with real values) we mean a function which vanishes outside a finite union of disjoint 'intervals' and is constant on each of them. Thus $f(\mathbb{R})$ is finite – the function only takes finitely many values – and

$$f^{-1}(c) \text{ is a finite union of disjoint intervals, } c \neq 0. \tag{4.3}$$

It is also often convenient to write a step function as a sum

$$f = \sum_{i=1}^{N} c_i \chi_{[a_i, b_i)} \tag{4.4}$$

of multiples of the characteristic functions of our intervals. Note that such a 'presentation' is not unique but can be made so by demanding that the intervals be disjoint and 'maximal' – so f is does not take the same value on two intervals with a common endpoint.

Now, a constant multiple of a step function is a step function and so is the sum of two step functions – clearly the range is finite. Really this reduces to checking that the difference $[a, b) \setminus [a', b')$ and the union of two intervals is always a union of intervals. The absolute value of a step function is also a step function.

A similar argument shows that the integral, defined by

$$\int_{\mathbb{R}} f = \sum_i c_i (b_i - a_i) \tag{4.5}$$

from (4.4) is independent of the 'presentation' used to define it. It is of course equal to the Riemann integral which is one way of seeing that it is well-defined (but of course the result is much more elementary).

Proposition 1. *The step functions on $\mathbb{R}$ in the sense defined above form a normed space with the L^1 norm*

$$\|f\|_{L^1} = \int_{\mathbb{R}} |f|. \tag{4.6}$$

So, we will complete this space instead of the continuous functions – it is both more standard and a little easier. The fact that we can directly construct a 'concrete' completion is due to Mikusiński. Already at this stage we can define a Lebesgue integrable function, however we need to do some work to flesh out the definition.

Definition 3. A function $g : \mathbb{R} \longrightarrow \mathbb{C}$ is Lebesgue integrable if there exists an absolutely summable sequence of step functions f_n, i.e. satisfying

$$\sum_n \int |f_n| < \infty \tag{4.7}$$

such that

$$f(x) = \sum_n f_n(x) \ \forall \ x \in \mathbb{R} \text{ such that } \sum_n |f_n(x)| < \infty. \tag{4.8}$$

So, the definition is a little convoluted – there must *exist* a sequence of step functions the sum the integrals of the absolute values of which converges and such that the sum itself converges to the function but only at the points of absolute converge of the (pointwise) series. Tricky, but one can get used to it – and ultimately simplify it. The main attraction of this definition is that it is self-contained and in principle 'everything' can be deduced from it.

Now, the first thing we need is the covering lemma – basically some properties of countable collections of our 'intervals' such as will arise when we look at a sequence, or series, of step functions. I leave it to you to devise careful proofs of the following two facts.

Lemma 1. *If $C_i = [a_i, b_i)$, $i = 1, \dots, N$, is a finite collection of intervals then*

$$C_i \subset [a, b) \ \forall \ i \text{ and } C_i \cap C_j = \emptyset \ \forall \ i \neq j \Longrightarrow \sum_{i=1}^{N} (b_i - a_i) \leq (b - a). \tag{4.9}$$

On the other hand

$$[a, b) \subset \bigcup_{i=1}^{N} C_i \Longrightarrow \sum_{i=1}^{N} (b_i - a_i) \geq (b - a). \tag{4.10}$$

You can prove this by inserting division points etc.

Now, what we want is the same thing for a countable collection of intervals.

Proposition 2. *If $C_i = [a_i, b_i)$, $i \in \mathbb{N}$, is a countable collection of intervals then*

$$C_i \subset [a, b) \ \forall \ i \text{ and } C_i \cap C_j = \emptyset \ \forall \ i \neq j \Longrightarrow \sum_{i=1}^{\infty} (b_i - a_i) \leq (b - a) \tag{4.11}$$

or

$$[a, b) \subset \bigcup_{i=1}^{N} C_i \Longrightarrow \sum_{i=1}^{\infty} (b_i - a_i) \geq (b - a). \tag{4.12}$$

Proof. You might think these are completely obvious, and the first is – the hypothesis (4.9) holds for any finite subcollection and hence the finite sum is always less than the fixed number $b - a$ and hence so is the infinite sum – which therefore converges.

On the other hand (4.12) is not quite so obvious since it depends on Heine-Borel. To be able to apply (4.10) choose $\delta > 0$. Now extend each interval by replacing the lower limit by $a_i - 2^{-i}\delta$ and consider the *open* intervals which therefore have the property

$$[a, b - \delta] \subset [a, b) \subset \bigcup_i (a_i - 2^{-i}\delta, b_i). \tag{4.13}$$

Now, by Heine-Borel – the compactness of closed bounded intervals – a finite subcollection of these open intervals covers $[a, b-\delta)$ so (4.10) *does* apply to the semi-open intervals and shows that for some finite N (hence including the finite subcollection)

$$\sum_{i=1}^{N}(b_i - a_i) - 2^{-i}\delta \geq b - a - \delta \Longrightarrow \sum_{i=1}^{\infty}(b_i - a_i) \geq b - a - 2\delta \tag{4.14}$$

where the sum here might be infinite, but then it is all the more true. The fact that this is true for all $\delta > 0$ now proves (4.12). □

Welcome to measure theory.

Okay, now the basis result on which most of the properties of integrable functions hinge is the following monotonicity lemma.

Lemma 2. *Let f_n be a sequence of step functions which decreases monotonically to* 0

$$f_n(x) \downarrow 0 \ \forall \ x \in \mathbb{R}, \tag{4.15}$$

then

$$\lim \int f_n = 0. \tag{4.16}$$

Note that 'decreases' here means that $f_n(x)$ is, for each x, a non-increasing sequence which has limit 0. Of course this means that all the f_n are non-negative. Moreover the first one vanishes outside some interval $[a, b)$ hence so do all of them. The crucial thing about this lemma is that we get the vanishing of the limit of the sequence of integrals without having to assume uniformity of the limit.

Proof. Going back to the definition of the integral of step functions, clearly $f_n(x) \geq f_{n+1}(x)$ for all x implies that $\int f_n$ is a decreasing (meaning non-increasing) sequence. So there are only two possibilities, it converges to 0, as we claim, or it converges to some positive value. This means that there is some $\delta >$ such that $\int f_n > \delta$ for all n, so we just need to show that this is not so.

Given an $\epsilon > 0$ consider the sets

$$S_j = \{x \in [a, b); f_j(x) \leq \epsilon\}. \tag{4.17}$$

(Here $[a, b)$ is an interval outside which f_1 vanishes and hence all the f_n vanish outside it). Each of the S_j is a finite union of intervals. Moreover

$$\bigcup_j S_j = [a, b) \tag{4.18}$$

since $f_n(x) \to 0$ for each x. In fact the S_j increase with j and if we set $B_1 = S_1$ and

$$B_j = S_{j+1} \setminus S_j \tag{4.19}$$

then the B_j are all disjoint, each consists of finitely many intervals and

$$\bigcup B_j = [a, b). \tag{4.20}$$

Thus, both halves of Proposition 2 apply to the intervals forming the B_j. If we let $l(B_j)$ be the length of B_j – the sum of the lengths of the finitely many intervals of which it is composed – then

$$\sum_j l(B_j) = b - a. \tag{4.21}$$

Now, let A be such that $f_1(x) \le A$ – and hence it is an upper bound for all the f_n's. We can choose N so large that

$$\sum_{j \ge N} l(B_j) < \epsilon. \tag{4.22}$$

Dividing the integral for f_k, $k \ge N$, into the part over S_N and the rest we see that

$$\int f_k \le (b-a)\epsilon + \epsilon A. \tag{4.23}$$

The first estimate comes for the fact the fact that $f_k \le \epsilon$ on S_N, and the second that the total lengths of the remaining intervals is no more than ϵ (and the function is no bigger than A.)

Thus the integral is eventually small! □

We can make this result look stronger as follows.

Proposition 3. *Let g_n be a sequence of real-valued step functions which is non-decreasing and such that*

$$\lim_{n\to\infty} g_n(x) \in [0,\infty] \ \forall\ x \in \mathbb{R} \tag{4.24}$$

then

$$\lim_{n\to\infty} \int g_n \in [0,\infty]. \tag{4.25}$$

I have written out $[0,\infty]$ on the right here to make sure that it is clear that we only demand that the non-decreasing sequence $g_n(x)$ 'becomes 0 or positive' in the limit – including the possibility that it 'converges' to $+\infty$ and then the same is true of the sequence of integrals.

Proof. Consider the functions

$$f_n(x) = \max(0, -g_n(x)) \ \forall\ x \in \mathbb{R}. \tag{4.26}$$

This is a sequence of non-negative step functions which decreases to 0 at each point and

$$\int g_n \ge -\int f_n. \tag{4.27}$$

Thus, as claimed, the result follows from the lemma. □

Next time we will use this to show that the definition of a Lebesgue integrable function above makes some sense. In particular we will show that the integral is well-defined!