18.102 Introduction to Functional Analysis
Spring 2009

Lecture 5. Thursday, 19 Feb

I may not get quite this far, since I do not want to rush unduly!

Let me denote by $\mathcal{L}^1(\mathbb{R})$ the space of Lebesgue integrable functions on the line – as defined last time.

Proposition 4. *$\mathcal{L}^1(\mathbb{R})$ is a linear space.*

Proof. The space of all functions on $\mathbb{R}$ is linear, so we just need to check that $\mathcal{L}^1(\mathbb{R})$ is closed under multiplication by constants and addition. The former is easy enough – multiplication by 0 gives the zero function which is integrable. If $g \in \mathcal{L}^1(\mathbb{R})$ then by definition there is an absolutely summable series of step function with elements f_n such that

$$\sum_n \int |f_n| < \infty,\ f(x) = \sum_n f_n(x)\ \forall\ x \text{ s.t. } \sum_n |f_n(x)| < \infty. \tag{5.1}$$

Then if $c \neq 0$, cf_n 'works' for cg.

The sum of two functions g and $g' \in \mathcal{L}^1(\mathbb{R})$ is a little trickier. The 'obvious' thing to do is to take the sum of the series of step functions. This will lead to trouble! Instead, suppose that f_n and f'_n are series of step functions showing that $g,\ g' \in \mathcal{L}^1(\mathbb{R})$. Then consider

$$h_n(x) = \begin{cases} f_k & n = 2k-1 \\ f'_k & n = 2k. \end{cases} \tag{5.2}$$

This is absolutely summable since

$$\sum_n \int |h_n| = \sum_k \int |f_k| + \sum_k \int |f'_k| < \infty. \tag{5.3}$$

More significantly, the series $\sum\limits_n |h_n(x)|$ converges if and only if *both* the series $\sum\limits_k |f_k(x)|$ and the series $\sum\limits_k |f'_k(x)$ converge. Then, because absolutely convergent series can be rearranged, it follows that

$$\sum_n |h_n(x)| < \infty \Longrightarrow \sum_n h_n(x) = \sum_k f_k(x) + \sum_k f'_k(x) = g(x) + g'(x). \tag{5.4}$$

Thus, $g + g' \in \mathcal{L}^1(\mathbb{R})$. □

So, the message here is to be a bit careful about the selection of the 'approximating' absolutely summable series. Here is another example.

Definition 4. A set $E \subset \mathbb{R}$ is *of measure zero* if there exists an absolutely summable series of step functions f_n such that

$$\sum_n |f_n(x)| = \infty\ \forall\ x \in E. \tag{5.5}$$

Proposition 5. *If $g \in \mathcal{L}^1(\mathbb{R})$ and $g' : \mathbb{R} \longrightarrow \mathbb{C}$ is such that $g' = g$ on $\mathbb{R} \setminus E$ where E is of measure zero, then $g' \in \mathcal{L}^1(\mathbb{R})$.*

Proof. What do we have to play with here – an absolutely summable series f_n of step functions which approximates g and another one, f'_n which is such that

$\sum\limits_n |f'_n(x)| = \infty$ for all $x \in E$. So, to approximate g' consider the interlaced series with terms

$$h_n(x) = \begin{cases} f_k(x) & n = 3k-2 \\ f'_k(x) & n = 3k-1 \\ -f'_n(x) & n = 3k. \end{cases} \tag{5.6}$$

Thus we add, but then subtract, f'_k. This series is absolutely summable with

$$\sum_n \int |h_n| = \sum_k \int |f_k| + 2\sum_k \int |f'_k|. \tag{5.7}$$

When does the pointwise series converge absolutely? We must have

$$\sum_k |f_k(x)| + 2\sum_k |f'_k(x)| < \infty. \tag{5.8}$$

The finiteness of the second term implies that $x \notin E$ and the finiteness of the first means that

$$\sum_n h_n(x) = g(x) = g'(x) \text{ when (5.8) holds} \tag{5.9}$$

since the finite sum is always $\sum\limits_{k=1}^{N} f_k(x)$, or this plus $f'_N(x)$ – which tends to zero with N by the absolute convergence of the *series* in (5.8). So indeed $g' \in \mathcal{L}^1(\mathbb{R})$. □

This certainly makes one conclude that sets of measure zero are small, except that we have not yet shown that $\mathcal{L}^1(\mathbb{R})$ really makes any sense. That it does starts to become clear when we check:

Proposition 6. *For any element $f \in \mathcal{L}^1(\mathbb{R})$ the integral*

$$\int f = \sum_n \int f_n \tag{5.10}$$

is well-defined independent of which approximating absolutely summable series of step functions satisfying (5.1) *is used to define it.*

Proof. We can suppose that f_n and f'_n are two absolutely summable series as in (5.1). Now that we have a little experience, it is probably natural to look at

$$h_n(x) = \begin{cases} f_k(x) & n = 2k-1 \\ -f'_k(x) & n = 2k. \end{cases} \tag{5.11}$$

This is absolutely summable and the pointwise series is absolutely convergent only when *both* series are absolutely convergent. The individual terms then tend to zero and so we see that

$$\sum_n |h_n(x)| < \infty \Longrightarrow \sum_n h_n(x) = 0. \tag{5.12}$$

Moreover, from the absolutely convergence of the sequence of integrals –

$$\sum_n |\int h_n| \le \sum_n \int |h_n| < \infty \tag{5.13}$$

it follows that we can rearrange the series to see that

$$\sum_n \int h_n = \sum_k \int f_k - \sum_k \int f'_k. \tag{5.14}$$

Now, what we want is that these two sums are equal, so we want to see that the left side of this equality vanishes. This follows directly from the next result which is just a little more general than needed here so is separated off. □

Proposition 7. *If an absolutely summable series of step functions satisfies* (5.1) *with* $f \equiv 0$ *then*

$$\sum_n \int f_n = 0. \tag{5.15}$$

Proof. So, the only thing we have at our disposal is the monotonicity result from last time. The trick is to use it! The 'trick' is to choose and $N \in \mathbb{N}$ and consider the new series of step functions with terms

$$g_1(x) = \sum_{j=1}^{N} f_j(x),\ \ g_k(x) = |f_{N+k-1}(x)|,\ \ k > 1. \tag{5.16}$$

Now, this is absolutely summable, since convergence is a property of the 'tail' and in any case

$$\sum_k \int |g_k| \le \sum_n \int |f_n|. \tag{5.17}$$

Moreover, since all the terms after the first are non-negative, the partial sums of the series

$$G_p(x) = \sum_{k=1}^{p} g_p(x) \text{ is non-decreasing.} \tag{5.18}$$

It is again a sequence of step functions. Note that there are two possibilities, depending on x. If the original series $\sum\limits_n |f_n(x)|$ diverges, i.e. converges to $+\infty$, then the same is true of G_p – since this is also a property of the tails. On the other hand, if $\sum\limits_n |f_n(x)|$ is finite, then for large p,

$$G_p(x) = \sum_{k=1}^{N} f_k(x) + \sum_{j=1}^{p-1} |f_{N+j}(x)| \ge \sum_{k=1}^{p+N-1} f_k(x). \tag{5.19}$$

The right side converges to zero, so the limit of this series (which is finite) is non-negative. So, the monotonicity proposition from last time applies to G_p and shows that

$$\lim_{p\to\infty} \int G_p \ge 0 \tag{5.20}$$

where divergence to $+\infty$ is a possibility. This however means that, for the N we originally chose,

$$\sum_{j=1}^{N} \int f_k + \sum_{k=1}^{\infty} \int |f_{N+k}| \ge 0. \tag{5.21}$$

This then is true for every N. On the other hand, the series of integrals is finite, so given $\delta > 0$ there exists M such that if $N > M$,

$$\sum_{k\ge M}\int|f_k| < \delta \Longrightarrow \sum_{j=1}^{N}\int f_k \ge -\delta \;\forall\; N > M. \tag{5.22}$$

This then implies that

$$\sum_k \int f_k \ge 0. \tag{5.23}$$

This is half of what we want, but the other half follows by applying the same reasoning to $-f_k$.

This the Proposition is proved. ☐

So, this is a pretty fine example of measure-integration reasoning.

Corollary 1. *The integral is a well-defined map*

$$\int : \mathcal{L}^1(\mathbb{R}) \longrightarrow \mathbb{C} \tag{5.24}$$

defined by setting

$$\int f = \sum_n \int f_n \tag{5.25}$$

for any approximating sequence as in (5.1).

In particular this integral is not trivial. Namely, if f is actually a step function then the sequence $f_1 = f$, $f_j = 0$ for all $j > 1$ is absoutely summable and approximates f in the sense of (5.1) so

$$\int f \text{ is consistent with the integral on step functions.} \tag{5.26}$$

So, we must be onto *something* here!

It seems I was pretty carrie away today – no doubt because there were not enough questions to slow me own. Hence I even went as far as to prove:-

Proposition 8. *A countable union of sets of measure zero has measure zero.*

Proof. By definition, a set E is of measure zero if there exists an absolutely summable series of step functions, f_n, so $\sum\limits_n \int|f_n| < \infty$ such that

$$\sum_n |f_n(x)| = +\infty \text{ on } E. \tag{5.27}$$

So, the data here gives us a countable collection E_j, $j = 1, \dots,$ of sets and for each of them we have an absolutely summable series of step functions $f_n^{(j)}$ such that

$$\sum_n \int |f_n^{(j)}| < \infty,\; E_j \subset \{x \in \mathbb{R}; \sum_n |f_n^{(j)}(x)| = +\infty\}. \tag{5.28}$$

The idea is to look for *one* series of step functions which is absolutely summable and which diverges absolutely on each E_j. The trick is to first 'improve' the $f_n^{(j)}$. Namely, the divergence in (5.28) is a property of the 'tail' – it persists if we toss out any finite number of terms. The absolute convergence means that for each j we can choose an N_j such that

$$\sum_{n\ge N_j}\int|f_n^{(j)}|<2^{-j}\ \forall\ j. \tag{5.29}$$

Now, simply throw away the all the terms in $f_n^{(j)}$ before $n=N_j$. If we relabel this new sequence as $f_n^{(j)}$ again, then we still have (5.28) but in addition we have not only absolute summability but also

$$\sum_n\int|f_n^{(j)}|\le 2^{-j}\ \forall\ j\Longrightarrow\sum_j\sum_n\int|f_n^{(j)}|<\infty. \tag{5.30}$$

Thus, the double sum (of integrals of absolutely values) is absolutely convergent.

Now, let h_k be the $f_n^{(j)}$ ordered in some reasonable way – say by working along each row $j+n=p$ in turn – in fact any enumeration of the double sequence will work. This is an absolutely summable series, because of (5.30). Moreover the pointwise series

$$\sum_k|h_k(x)|=+\infty \text{ if } \sum_n|f_n^{(j)}(x)|=+\infty \text{ for any } j \tag{5.31}$$

since the second sum is contained in the first. Thus $\sum\limits_k|h_k(x)|$ diverges at each point of each E_j, so the union has measure zero. □