18.103 Fall 2013

1 Fourier Integrals on $L^2(\mathbb{R})$ and $L^1(\mathbb{R})$.

The first part of these notes cover §3.5 of AG, without proofs. When we get to things not covered in the book, we will start giving proofs.

The Fourier transform is defined for $f \in L^1(\mathbb{R})$ by

$$\mathcal{F}(f) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi}\, dx \tag{1}$$

The Fourier inversion formula on the Schwartz class $\mathcal{S}(\mathbb{R})$.

Theorem 1 *If $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$ and*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi$$

Thus the inverse operator to the Fourier transform is given by

$$\check{g}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi) e^{ix\xi} d\xi = \frac{1}{2\pi} \hat{g}(-x)$$

A function $f \in L^2(\mathbb{R})$ need not be in $L^1(\mathbb{R})$ and the integral defining $\hat{f}$ may be divergent. Nevertheless, one can define the Fourier transform $\hat{f}$ as a limit in two ways. The first way uses the Plancherel theorem.

Corollary 1 *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$2\pi \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi$$

Corollary 1 leads to a definition of the Fourier transform for $f \in L^2(\mathbb{R})$ by continuity in the L^2 distance as follows.

Corollary 2 *Let $f \in L^2(\mathbb{R})$ and let $f_j \in \mathcal{S}(\mathbb{R})$ be such that $\|f - f_j\|_2 \to 0$ as $j \to \infty$. Then $\hat{f}_j$ is a Cauchy sequence in $L^2(\mathbb{R})$ and the limit (in the L^2 metric) is independent of the choice of sequence approximating f. Thus there is a unique function denoted $\hat{f} \in L^2(\mathbb{R})$ for which*

$$\lim_{j\to\infty} \|\hat{f} - \hat{f}_j\|_2 = 0$$

Furthermore,

$$\|\hat{f}\|_2^2 = 2\pi \|f\|_2^2$$

Corollary 3 *If $f \in L^2(\mathbb{R})$, and $\hat{f} = 0$ almost everywhere, then $f = 0$.*

Fourier inversion formula on the Schwartz class extends by continuity to Fourier inversion on $L^2(\mathbb{R})$.

Corollary 4 *(Fourier inversion on L^2) Let*

$$\mathcal{G}(f)(x) = \frac{1}{2\pi}\hat{f}(-x)$$

then for all $f \in L^2(\mathbb{R})$,

$$\mathcal{G} \circ \mathcal{F}(f) = \mathcal{F} \circ \mathcal{G}(f) = f$$

Thus, up to the factor 2π, the Fourier transform is an isometry (distance preserving) from $L^2(\mathbb{R})$ to itself.

We need to make sure that our two definitions of the Fourier transform for L^1 and L^2 are consistent. This is taken care of by the following proposition.

Proposition 1 *If $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then the definition by continuity in Corollary 2 for $\hat{f}$ coincides with the definition* (1) *above.*

(See PS9, Exercise AG §3.5/3, p. 153. The starting point of the proof of the proposition is that one can choose $f_j \in \mathcal{S}$ so that $\|f - f_j\|_1 + \|f - f_j\|_2 \to 0$).

As a consequence of the proposition, we find a second way to define the Fourier transform on L^2 using a more straightforward truncation Indeed, in the very next exercise (PS9, AG §3.5/4, p. 153) you were asked to show that if $f \in L^2(\mathbb{R})$, then

$$\hat{f}(\xi) = \lim_{N\to\infty} \int_{-N}^{N} f(x) e^{-ix\xi}\, dx, \qquad \text{(limit in } L^2 \text{ sense)}$$

To prove this, note that if $f_N(x) = f(x)1_{[-N,N]}$, then $f_N \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, and by Exercise §3.5/3, $\hat{f}_N(\xi)$ is the integral on the right. On the other hand, it follows from Corollary 2 applied to $f - f_N$ that

$$\|\hat{f} - \hat{f}_N\|_2^2 = 2\pi\|f - f_N\|_2^2 = 2\pi \int_{|x|>N} |f(x)|^2 dx$$

which tends to zero by the dominated convergence theorem (with majorant $|f(x)|^2$).

We now deduce a more explicit version of Fourier inversion on L^2, which can be stated as follows.

Theorem 2 *Suppose that $f \in L^2(\mathbb{R})$. Then $\hat{f}(\xi)1_{[-N,N]}(\xi)$ is in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and*

$$s_N(x) = \frac{1}{2\pi} \int_{-N}^{N} \hat{f}(\xi) e^{ix\xi} d\xi$$

satisfies

$$\lim_{N\to\infty} \|f - s_N\|_{L^2} = 0$$

To begin the proof of Theorem 2, consider $f \in L^2(\mathbb{R})$. Then by Corollary 2, $\hat{f} \in L^2(\mathbb{R})$ and hence, by the Cauchy-Schwarz inequality, $\hat{f}\, 1_{[-N,N]} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. We will apply a proposition analogous to Proposition 1 (with exactly the same proof).

Proposition 2 *If $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the inverse Fourier tranform obtained by continuity in the L^2 norm coincides with the L^1 definition:*

$$\mathcal{G}(h)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(\xi) e^{ix\xi} d\xi$$

Let $h = \hat{f}\, 1_{[-N,N]}$. Then $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and Proposition 2 implies

$$s_N(x) = \mathcal{G}(h)(x)$$

Since $h \in L^2(\mathbb{R})$, we also have $s_N \in L^2(\mathbb{R})$, and we may take the Fourier transform and apply Theorem 4 to obtain

$$\hat{s}_N(\xi) = h(\xi) = \hat{f}(\xi)1_{[-N,N]}(\xi)$$

Finally, applying the formula in Corollary 2

$$2\pi\|f - s_N\|_2^2 = \|\hat{f} - \hat{s}_N\|_2^2 = \int_{|\xi|>N} |\hat{f}(\xi)|^2 d\xi \to 0 \quad \text{as} \quad N \to \infty$$

(The last step uses the dominated convergence theorem with majorant $|\hat{f}(\xi|^2$.) This ends the proof of Theorem 2.

Our last task is to find a Fourier inversion formula on $L^1(\mathbb{R})$.

Theorem 3 *Let $f \in L^1(\mathbb{R})$ and denote*

$$\sigma_N(x) = \frac{1}{2\pi} \int_{-N}^{N} (1 - |\xi/N|)^+ \hat{f}(\xi) e^{ix\xi} d\xi$$

Then

$$\lim_{N\to\infty} \|f - \sigma_N\|_{L^1} = 0$$

Corollary 5 *If $f \in L^1(\mathbb{R})$, and $\hat{f} = 0$, then $f = 0$.*

The idea of the proof of Theorem 3 is parallel to the case of Fourier series. Note that Fubini's theorem implies that for f and g in $L^1(\mathbb{R})$,

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \tag{2}$$

We will show that

$$\sigma_N(x) = f * F_N(x) \tag{3}$$

for a function F_N, known (as in the the case of the circle group) as the Fejér kernel.

Theorem 4 *(Approximate identity) If $K \in L^1(\mathbb{R})$, $K_\epsilon(x) = (1/\epsilon)K_\epsilon(x)$, and*

$$\int_{-\infty}^{\infty} K(x)\, dx = 1$$

then $\|K_\epsilon * f - f\|_1 \to 0$ for all $f \in L^1(\mathbb{R})$.

Consider $K(x) = F_1(x)$, $K_\epsilon = F_{1/\epsilon}$ with $\epsilon = 1/N$. It will suffice to show that $K = F_1$ is integrable with integral 1. In fact, we will find that $F_1(x) > 0$ and

$$\int_{\mathbb{R}} |F_1(x)| dx = \int_{\mathbb{R}} F_1(x) dx = \hat{F}_1(0) = 1 \tag{4}$$

Thus the approximate identity theorem implies that $\|\sigma_N - f\|_1 \to 0$ as $N \to \infty$ for all $f \in L^1(\mathbb{R})$.

We will find the formula for F_N using the identity

$$\hat{F}_N(\xi) = (1 - |\xi/N|)^+$$

This function has the shape of a triangle. It has a very simple relationship with change of scale, namely, $\hat{F}_N(\xi) = \hat{F}_1(\xi/N)$ and by change of variables, $F_N(x) = NF_1(Nx)$. One can easily compute F_1 and hence F_N using the inverse Fourier transform formula and integration by parts, but we prefer to derive its fomula by a more circuitous route that will enable us to see why $F_N(x)$ is essentially the square of $D_N(x)$, the Dirichlet kernel.

Define

$$\hat{D}_N(\xi) = 1_{[-N,N]}(\xi)$$

Then Proposition 2 gives

$$D_N(x) = \frac{1}{2\pi} \int_{-N}^{N} e^{ix\xi}\, d\xi = \left. \frac{e^{ix\xi}}{2\pi i x} \right|_{-N}^{N} = \frac{\sin Nx}{\pi x}$$

D_N is known as the Dirichlet kernel (analogous to the one for Fourier series).

$$s_N(x) = f * D_N(x); \quad \hat{s}_N(\xi) = \hat{f}(\xi) 1_{[-N,N]}(\xi)$$

(Note that $D_N(x)^2 \le 1/|x|^2$ as $|x| \to \infty$ so that $D_N \in L^2(\mathbb{R})$. Thus $f * D_N(x)$ is a convergent integral for every x, provided $f \in L^2(\mathbb{R})$.) We also remark that D_N has the following scaling properties.

$$D_N(x) = ND_1(Nx); \quad \hat{D}_N(\xi) = \hat{D}_1(\xi/N)$$

As in the case of Fourier series, it does not work to approximate f by $s_N(x)$ for $f \in L^1(\mathbb{R})$. By inspection, we see that $|D_N(x)|$ has the size of $1/|x|$ as $|x| \to \infty$ so that $D_N \notin L^1(\mathbb{R})$. Even figuring out exactly what $f * D_N(x)$ means for $f \in L^1(\mathbb{R})$ is delicate and beyond the scope of this course.

So instead of D_N, we work out the formula for the Fejér kernel F_N. Since $\hat{D}_{1/2}(\xi) = 1_{[-1/2,1/2]}$, we have the convolution formula

$$\hat{D}_{1/2} * \hat{D}_{1/2}(\xi) = (1 - |\xi|)^+ = \hat{F}_1(\xi)$$

Because the inverse Fourier transform is $1/2\pi$ times the Fourier transform (with a sign change) a formula equivalent to (2) says

$$\mathcal{G}(f * g) = 2\pi\mathcal{G}(f)\mathcal{G}(g)$$

Apply this with $f = g = 1_{[-1/2,1/2]} = \hat{D}_{1/2}$, then

$$F_1 = \mathcal{G}(f * g) = 2\pi\mathcal{G}(f)\mathcal{G}(g) = 2\pi D_{1/2}^2$$

In other words,

$$F_1(x) = 2\pi\frac{\sin^2(x/2)}{(\pi x)^2} = \frac{2\sin^2(x/2)}{\pi x^2}$$

Next we rescale. Since $\hat{F}_N(\xi) = \hat{F}_1(\xi/N)$, we have

$$F_N(x) = NF_1(Nx) = \frac{2N\sin^2(Nx/2)}{\pi(Nx)^2} = \frac{2\sin^2(Nx/2)}{\pi Nx^2}$$

The only feature of the explicit formula for $F_N(x)$ that we need is $F_N(x) > 0$. Since $\hat{F}_N(0) = 1$, (4) follows.

The last step in the proof is to confirm (3). If $f \in L^1(\mathbb{R})$, then $\hat{f}$ is continuous and by definition,

$$\sigma_N(x) = \frac{1}{2\pi}\int_{-N}^{N} \hat{f}(\xi)(1 - |\xi/N|)e^{ix\xi}\, d\xi = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(y)e^{-iy\xi}\, dy\, (1 - |\xi/N|)e^{ix\xi}\, d\xi$$

The majorant $|f(y)|(1 - |\xi/N|)^+$ is integrable with respect to $dyd\xi$ so Fubini's theorem and Theorem 2 applied to F_N imply

$$\sigma_N(x) = \int_{\mathbb{R}} f(y)\frac{1}{2\pi}\int_{\mathbb{R}}(1 - |\xi/N|)^+ e^{i(x-y)\xi} d\xi\, dy = f * F_N(x)$$

As a final remark, we double check our arithmetic in the computation of F_N as follows.

$$F_N(0) = \frac{1}{2\pi}\int_{-N}^{N}(1 - |\xi/N|)\, d\xi$$

The integral on the right is $1/2\pi$ times the area of the triangle of base $2N$ and height 1, so the total is $N/2\pi$. The left side is

$$F_N(0) = \lim_{x\to 0} \frac{2\sin^2(Nx/2)}{\pi N x^2} = \lim_{x\to 0} \frac{2(Nx/2)^2}{\pi N x^2} = N/2\pi$$