18.117 Assignment # 4

1. Let $\Delta \subseteq \mathbb{R}^d$ be a simple m-dimensional convex polytope, and let $\xi \in \mathbb{R}^d$. Assume that $\langle \xi, v' - v \rangle \neq 0$ for every pair of adjacent vertices v and v' of Δ.

   For $v \in \mathrm{Vert}(\Delta)$, define
$$\mathrm{ind}_\xi v = \#\{v_i, \langle v_i - v, \xi \rangle < 0\},$$
   where $v_1, ..., v_m$ are the vertices adjacent to v.

   Prove that
$$b_k \equiv \#\{v \in \mathrm{Vert}(\Delta), \mathrm{ind}_\xi v = k\}$$
   is independent of ξ.

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   Hints for Problem 1:

   (a) Let F be a k-dimensional face of Δ. Show that there is a unique vertex v_F $\in$Vert(F) such that if $v_1, ..., v_k$ are the vertices of F adjacent to v_F, then
$$\langle v_i - v_F, \xi \rangle > 0, \quad (*)$$
   for all $i = 1, ..., k$.

   (b) In particular, if $F = \Delta$, then there is *one* vertex v_0 with the property $(*)$. Conclude that $b_0 = 1$.

   (c) Let f_{m-1} be the number of $(m-1)$-dimensional faces of Δ. Each such face F has a unique vertex v_F with property $(*)$. Show that there are m such faces with $v_F = v_0$, and show that the number of such faces with $v_F \neq v_0$ is b_1. Conclude that $f_{m-1} = mb_0 + b_1$.

   (d) Let f_{m-2} be the number of $(m-2)$-dimensional faces of Δ. Show that
$$f_{m-2} = \binom{m}{2} b_0 + \binom{m-1}{1} b_1 + b_2.$$

   i. Sub-hint: The first summand counts the number of $(m-2)$-dimensional faces F for which $v_F = v_0$.
   The second summand counts the number of $(m-2)$-dimensional faces F for which $v_F = v_{F'}$, where F' is an $(m-1)$-dimensional face with $v_{F'} \neq v_0$.
   The third summand counts all the other $(m-2)$-dimensional faces.

   (e) In general, conclude that
$$f_{m-k} = \binom{m}{k} b_0 + \binom{m-1}{k-1} b_1 + ... + b_k = \sum_{\ell=0}^{k} \binom{m-\ell}{k-\ell} b_\ell.$$