Lecture 23

For the next few days we're assuming that B is symplectic and $V = V^{2n}$. Choose a Darboux basis $e_1, f_1, \dots, e_n, f_n$. Check that $L_B : V \to V^*$ is the map

$$\{e_i \to -f_i^*, f_i \to e_i^*\}$$

where e_i^*, f_i^* are the dual vectors. In the symplectic case $B^\sharp = -B$ and $L_{B^\sharp} = -L$.

Say that $\omega \in \Lambda^2 V$,

$$\omega = \sum e_i \wedge f_i$$

Then we have the operation $L : \Lambda^p \to \Lambda^{p+2}$, given by $\alpha \mapsto \omega \wedge \alpha$ and also its transpose $L^t : \Lambda^{p+2} \to \Lambda^p$. Lets look at the commutator $[L, L^t] : \Lambda^p \to \Lambda^p$.

Theorem (Kaehler, Weil). $[L, L^t] = (p - n)\,\mathrm{Id}$

Proof. $L = \sum_i L_{e_i} L_{f_i}$, so

$$L^t = \sum_i L^t_{f_i} L^t_{e_i} = \sum \iota_{f_i^*} \iota_{e_i^*}$$

Its easy to see that Kaehler-Weil holds when $n = 2$.

For n-dimensions

$$L = \sum L_i \qquad L_i = L_{e_i} L_{f_i} \qquad L^t = \sum L^t_i \qquad L^t_i = \iota_{f_i^*} \iota_{e_i^*}$$

$V_i = span\{e_i, f_i\}$, then $\Lambda^p = span \beta_1 \wedge \cdots \wedge \beta_n$ where $\beta_i \in \Lambda^{p_i}(V_i)$.

Note that

$$L_i \beta_1 \wedge \cdots \wedge \beta_n = \beta_1 \wedge \cdots \wedge (L_i \beta_i) \wedge \cdots \wedge \beta_n$$

and

$$L^t_j(\beta_1 \wedge \cdots \wedge \beta_n) = \beta_1 \wedge \cdots \wedge (L_j \beta_j) \wedge \cdots \wedge \beta_n$$

If $n \neq j$, then $L_i L^t_j = L^t_j L_i$. So

$$\begin{aligned}[L, L^t]\beta_1 \wedge \cdots \wedge \beta_n &= \sum_i \beta_1 \wedge \cdots \wedge [L_i, L^t_i]\beta_i \wedge \cdots \wedge \beta_n \\ &= \sum^n (p_i - 1)\beta_1 \wedge \cdots \wedge \beta_n = (p - n)\beta_1 \wedge \cdots \wedge \beta_n\end{aligned}$$

□