Lecture 33

First, some general Lie theory things. G a compact, connected Lie group. Let $G_{\mathbb{C}} \supset G$ a complex Lie group.

Definition. $G_{\mathbb{C}}$ is the complexification of G if

(a) $\mathfrak{g}_{\mathbb{C}} = Lie\ G_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$

(b) The complex structure on $T_e G_{\mathbb{C}}$ is the standard complex structure on $\mathfrak{g} \otimes \mathbb{C}$.

(c) $\exp : \mathfrak{g}_{\mathbb{C}} \to G_{\mathbb{C}}$ maps $\mathfrak{g}$ into G.

(d) The map $\sqrt{-1}\mathfrak{g} \times G \to G_{\mathbb{C}}$ defined by $(\omega, g) \mapsto (\exp \omega) g$ is a diffeomorphism.

Take $G = U(n)$. What is $\mathfrak{g}$? Let H_n be the Hermitian matrices. If $A \in H_n$, then $\exp \sqrt{-1} t A \subset U(n)$, so $\mathfrak{g} = \sqrt{-1} H_n$.

Exercise Show $G_{\mathbb{C}} = GL(n, \mathbb{C})$

Hints:

(a) $M_n(\mathbb{C}) = Lie\ GL(n, \mathbb{C}) = H_n \oplus \sqrt{-1} H_n$ given by the decomposition

$$A \mapsto \frac{A + \bar{A}^t}{2} + \frac{A - \bar{A}^t}{2}$$

(b) Polar decomposition theorem: For $A \in GL(n, \mathbb{C})$ then $A = BC$ where B is positive definite, $B \in H^n$ and $C \in U(n)$.

(c) $\exp : H_n^* \to H_n^{*\text{pos. def}}$ is an isomorphism. This maps a matrix with eigenvalues λ_i to a matrix with eigenvalues e^{λ_i}.

Example. Take G a compact, connected abelian Lie group. Then $G = \mathfrak{g}/\mathbb{Z}_G$ and $G_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}/\mathbb{Z}_G$.

Let M be a Kaehler manifold, ω a Kaehler form, and τ a holomorphic action of $G_{\mathbb{C}}$ on M.

Definition. τ is a **Kaehler action** if $\tau\ |_G$ is hamiltonian.

So we have a moment map $\Phi : M \to \mathfrak{g}^*$ and for $v \in \mathfrak{g}$ we have v_M a vector field on M, and

$$\iota(v_M)\omega = d\phi^v \qquad \phi^v = \langle \Phi, v \rangle$$

For $p \in M$ note that because M is Kaehler we have the addition bits of structure $(B_r)_p, (B_s)_p, J_p$ on $T_p M$.

Now take $v \in \mathfrak{g}$, $\sqrt{-1} v = w \in \mathfrak{g}_{\mathbb{C}}$. From these we get corresponding vector fields v_M, w_M.

Lemma. *At every* $p \in M$

$$w_M(p) = J_p v_M(p)$$

Proof. Consider $\epsilon : G_{\mathbb{C}} \to M$, $g \mapsto \tau_{g^{-1}}(p)$. This is a holomorphic map and $(d\epsilon)_p : \mathfrak{g}_{\mathbb{C}} \to T_p M$ is $\mathbb{C}$-linear and maps v, w into $v_M(p), w_M(p)$.

Proposition. *If* $v \in \mathfrak{g}$, $w = \sqrt{-1} v$, *ten the vector field* w_M *is the* *Riemannian* *gradient of* ϕ^v.

Proof. Take $p \in M$, $v \in T_p M$. Then

$$(B_r)_p(v, w_M(p)) = B_s(v, J_p w_M(p)) = -B_s(v, v_M(p)) = \iota(v_M(p))\omega_p(v) = d\phi^v_p(v)$$

QED ∎

Assume $\Phi : M \to \mathfrak{g}^*$ is proper. Let $Z = \Phi^{-1}(0)$. Assume that G acts freely on Z. Then Z is a compact submanifold of M. Then we can form the reduction $M_{red} = Z/G$.

Consider $G_{\mathbb{C}} \times Z \to M$ given by $(g, z) \mapsto \tau_g(z)$. Let M_{st} be the image of this map. Note that $G_{\mathbb{C}}$ is a subset of M.

Theorem (Main Theorem). *(a) M_{st} is an open $G_{\mathbb{C}}$-invariant subset of M.*

(b) $G_{\mathbb{C}}$ acts freely and properly on M_{st}.

(c) Every $G_{\mathbb{C}}$ orbit in M_{st} intersects Z in a unique G-orbit.

(d) Hence $M_{st}/G_{\mathbb{C}} = Z/G = M_{red}$.

(e) ω_{red} is Kaehler.

Proof. (a) Since M_{st} is $G_{\mathbb{C}}$-invariant it suffices to show that M_{st} contains an open neighborhood of Z. Note that since $G_{\mathbb{C}} = (\exp \sqrt{-1}g)G$ implies that M_{st} is the image of

$$\psi : \sqrt{-1}g \times Z \to M \qquad (\omega, p) \mapsto (\exp w_m)(p)$$

Hence it suffices to show that ψ is a local diffeomorphism at all points $(0, p)$. Hence it suffices to show that $(d\psi)_{0,p}$ is bijective.

But $(d\pi)_{0,p} : T_pZ \to T_pZ$. So it suffices to finally prove that

Lemma. *$(d\psi)_{0,p}$ maps $\sqrt{-1}\mathfrak{g}$ bijectively onto $(T_pZ)_p^{\perp}$ in T_pM.*

Proof. Let $w = \sqrt{-1}v$ in $\sqrt{-1}\mathfrak{g}$, $v \in T_pZ$. Then

$$B_r(v, w_M(p)) = d\varphi_p^v(v) = 0$$

so $w_M(p) \perp T_pZ$.

(b) $G_{\mathcal{C}}$ acts freely on M_{st}.

Lemma. *If $p \in Z$ and $w \in \sqrt{-1}\mathfrak{g} - \{0\}$. Then $(\exp w_M)(p) \in Z$.*

Proof. Let $w = \sqrt{-1}v, v \in \mathfrak{g}$, then $(\exp tw_M)(p)$ is an integral curve of a gradient vector field of φ^v. Now $\varphi^v(p) = 0$ so $\varphi^v(\exp tw_M)(p) > 0$ for $t > 0$ (since gradient vector fields are increasing. So $\varphi^v(\exp w_M)(p) > 0$ and so $\exp w_M(p) \notin Z$.

To show that $G_{\mathbb{C}}$ acts freely on M_{st} it suffices to show that $G_{\mathbb{C}}$ acts freely at $p \in Z$. Let $a \in G_{\mathbb{C}}$, $a = (\exp -w)_g$, where $w \in \sqrt{-1}\mathfrak{g}, g \in G$. Suppose $a \in (G_{\mathbb{C}})_p$ then $(\exp w_M)(\tau_g(p)) = p$. But $\tau_g(p) = q \in Z$. So $(\exp w_M)(q) = p \in Z$ which implies $w = 0$, $a = G$. So $(G_{\mathbb{C}}) = G_p = \{e\}$.

We will skip proving that $G_{\mathbb{C}}$ acts properly on M_{st}.

(c) This will be an exercise

Exercise Every $G_{\mathbb{C}}$-orbit in M_{st} intersects Z in a unique G orbit. Hint: Every $G_{\mathbb{C}}$ orbit in M_{st} is of the form $G_{\mathbb{C}} \circ p$ with $p \in Z$. $a \in (G_{\mathbb{C}} \circ p) \cap Z$. Then $a = (\exp w_M)\tau_g(p)$, $g \in G$, $w \in -sqrt{-}1\mathfrak{g}$. Argue as before and force $w = 0$.

(d) So $M_{red} = Z/G = M_{st}/G_{\mathbb{C}}$.

(e) All that remains to show is that ω_{red} is Kaehler.

Proof. $p \in Z$, $\pi : Z \to M_{red}$, $q = \pi(p)$. Let V be the B_r-orthocomplement in T_pM to $T_p(G_{\mathbb{C}} \circ p)$ implies that $V \subseteq T_pZ$ and its perpendicular to $T_pG \circ p$.

Remember we have $d\pi : M_{st} \to M_{red} = M_{st}/G_{\mathbb{C}}$ is a holomorphic action.

So $d\pi_p : V \to T_qM_{red}$ is $\mathbb{C}$-linear and $\omega_p \mid V = (d\pi_p)^*\omega_{red} \mid_V$, where V a complementary subspace of T_pM so $\omega_p \mid$ is Kaehler implies that $(\omega_{red})_q$ is Kaehler. $\square$