MEASURE AND INTEGRATION: LECTURE 2

Proposition 0.1. *Let $\mathcal{M}$ be a σ-algebra on X, let Y be a topological space, and let $f\colon X \to Y$.*

(a) *Let Ω be a collection of sets $E \subset Y$ such that $f^{-1}(E) \in \mathcal{M}$. Then Ω is a σ-algebra on Y.*
(b) *If f is measurable and $E \subset Y$ is Borel, then $f^{-1}(E) \in \mathcal{M}$.*
(c) *If $Y = [-\infty, \infty]$ (with open sets along with $[-\infty, a)$ and $(b, \infty]$ with $a, b \in \mathbb{R}$) and $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ for all $\alpha \in [-\infty, \infty]$, then f is measurable.*

Proof. (a) Since $f^{-1}(Y) = X \in \mathcal{M}$, we have $Y \in \Omega$. Also $f^{-1}(E^c) = (f^{-1}(E))^c \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$. Lastly,

$$f^{-1}(\cup_{i=1}^{\infty} E_i) = \cup_{i=1}^{\infty} f^{-1}(E_i) \in \mathcal{M}.$$

(b) Because f is measurable, all open sets are in Ω. Since Ω is a σ-algebra, we have $\mathcal{B} \subset \Omega$.
(c) Recall $\Omega = \{E \mid f^{-1}(E) \in \mathcal{M}\}$. Given $\alpha \in \mathbb{R}$, choose $\alpha_n < \alpha$ so that $\alpha_n \to \alpha$ as $n \to \infty$. By assumption $(\alpha_n, \infty] \in \Omega$. Then $[-\infty, \alpha) = \cup_{n=1}^{\infty} [-\infty, \alpha_n] = \cup_{n=1}^{\infty} (\alpha_n, \infty]^c$. Thus $[-\infty, \alpha) \in \Omega$. Then $(\alpha, \beta) = [-\infty, \beta) \cap (\alpha, \infty] \in \Omega$. Hence, since Ω is a σ-algebra, Ω contains all open sets. It follows that f is measurable.

$\square$

Remark. All of these are equivalent:

$$\begin{aligned} f^{-1}([-\infty, \alpha)) \in \mathcal{M} \quad &\iff f^{-1}([-\infty, \alpha]) \in \mathcal{M} \\ &\iff f^{-1}([\alpha, \infty]) \in \mathcal{M} \\ &\iff f^{-1}((-\infty, \alpha)) \;\&\; f^{-1}(\{-\infty\}) \in \mathcal{M}. \end{aligned}$$

Limits. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ or $[-\infty, \infty]$. Set

$$b_k = \sup\{a_k, a_{k+1}, \ldots\}, \;\; k = 1, 2, \ldots.$$

Then $\inf b_k = \limsup_{n\to\infty} a_n$. As k gets larger, the sup is being taken over a smaller set, so $b_k \geq b_{k+1} \geq \ldots$. Thus b_k is a (weakly) decreasing sequence and so $\lim b_k = \inf b_k$ exists. In other words,

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lim sup is the largest limit point of the sequence (there exists a subsequence which converges to lim sup). Similarly, we could instead set $b_k = \inf\{a_k, a_{k+1}, \ldots\}$ and then $\sup b_k = \liminf_{n\to\infty} a_n$, that is, lim inf is the smallest limit point of the sequence. Note the relation $\liminf a_n = -\limsup\{-a_n\}$.

Proposition 0.2. *A sequence $\{a_n\}$ converges if and only if*

$$\liminf a_n = \limsup a_n = \lim a_n.$$

Limits of functions. Let $\{f_n\} \colon X \to \mathbb{R}$ be a sequence of functions.

$$(\sup f_n)(x) = \sup\{f_n(x)\}$$

$$(\limsup f_n)(x) = \lim_{n\to\infty} f_n(x).$$

If, for each $x \in X$, the sequence $\{f_n(x)\}$ converges, then $f(x) = \lim f_n(x)$ is the pointwise limit. This works for $X = [-\infty, \infty]$ (convergence to $\pm\infty$ is obvious).

Theorem 0.3. *If, for each $i = 1, 2, \ldots$, the function $f_i \colon X \to [-\infty, \infty]$ is measurable, then*

$$g = \sup_{i>1} f_i \text{ and } h = \limsup_{n\to\infty} f_n$$

are both measurable.

Proof. NTS $g^{-1}((\alpha, \infty]) \in \mathcal{M}$ for all α. We have

$$g^{-1}((\alpha,\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty]).$$

If x is a member of the LHS, then $g(x) > \alpha$. Thus, some $f_n > \alpha$ from the definition of sup. If x is a member of the RHS, then $f_i(x) > \alpha$ for some i, so $g \geq f_i(x) > \alpha$. Thus, $x \in g^{-1}((\alpha,\infty])$. Since $f_n^{-1}((\alpha,\infty]) \in \mathcal{M}$, and since countable unions are in $\mathcal{M}$, $g \in \mathcal{M}$. But then lim sup is measurable as well since by definition

$$\limsup f_k = \inf_{j\geq 1}\left(\sup_{k\geq j} f_k\right).$$

□

Corollary 0.4. (a) *Pointwise limits of measurable functions are measurable.*
(b) *If f and g are measurable, then $\max\{f,g\}$ and $\min\{fmg\}$ are measurable.*

Define "f plus" and "f minus" as follows:

$$f^+ = \max\{f, 0\},\ f^- = -\min\{f, 0\}.$$

Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Simple functions. A *simple function* is a function that takes only finitely many values in $\mathbb{R}$ and does not take values $\pm\infty$. Let $\alpha_1, \dots, \alpha_n$ be the values and $A_i = \{x \in X \mid s(x) = \alpha_i\}$. Then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}.$$

Note: χ_{A_i} is measurable $\iff$ $A_i \in \mathcal{M}$. Constant function α_i is measurable $\Rightarrow$ product $\alpha_i \chi_{A_i}$ is measurable. Since sums of measurable functions are measurable, s is measurable $\iff$ all A_i are measurable.

Theorem 0.5. *Let $f\colon X \to [0, \infty]$ be measurable. Then there exists a sequence $0 \le s_1 \le s_2 \le \dots$ of measurable simple functions such that $\lim_{n\to\infty} s_n = f$.*

Proof. Partition $[0, n]$ into intervals of length 2^{-n}. Define $\varphi_n\colon [0, \infty] \to [0, \infty)$ as follows. Let $\delta_n = 2^{-n}$. For each t, choose $k_n(t)$ such that $k\delta_n \le t < (k + 1)\delta_n$. Put

$$\varphi_n(t) = \begin{cases} k_n(t)\delta_n & 0 \le t < n; \\ n & n \le t \le \infty. \end{cases}$$

Note that each φ_n is Borel measurable, $\varphi_1 \le \varphi_2 \le \cdots \le t$ and $\lim_{n\to\infty} \varphi_n(t) = t$. Let $s_n = \varphi_n \circ f$. Then for any open set U, $s_n^{-1}(U) = f^{-1}\varphi_n^{-1}(U) = f^{-1}(\text{Borel set}) \in \mathcal{M}$. Thus s_n is measurable, increasing, and its limit is f. □

A *positive measure* is a mapping $\mu\colon \mathcal{M} \to [0, \infty]$ which is countably additive:

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \text{ for disjoint } A_i.$$

Lebesgue Integral. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $s\colon X \to [0, \infty)$ be the simple function $s = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$, where the A_i are disjoint. For each $E \in \mathcal{M}$, define the integral

$$\int_E s\,d\mu = \sum_{i=1}^{N} \alpha_i \mu(A_i \cap E).$$

For more general measurable functions $f\colon X \to [0,\infty]$, then

$$\int_E f d\mu = \sup \left\{ \int_E s d\mu \mid s \text{ simple}, 0 \le s \le f \right\}.$$

If $f\colon X \to [-\infty,\infty]$, then

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu,$$

provided both terms on the right-hand side are finite.