Lecture 17: Holder continuity of L-Harmonic operators part I

Over the next few lectures we'll prove that L harmonic functions are Holder continuous. We'll consider operators in divergence form.

1 Operators in divergence form

Theorem 1.1 *Let L be a uniformly elliptic operator taking*

$$Lu = \frac{\partial}{\partial x_j} A_{ij} \frac{\partial u}{\partial x_j} \tag{1}$$

with $0 < \lambda |v|^2 \leq v \cdot (Av) \leq \Lambda |v|^2$ as usual, and let u be an L harmonic function. Then u is Holder continuous.

Eventually we will prove this via Morrey's lemma, but we need to do some simplification first. As before we define

$$A_{x,r} = \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} u, \tag{2}$$

and

$$(\nabla f)_{x,r} = \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} (\nabla f). \tag{3}$$

Given a point $x_0 \in \mathbb{R}^n$ we can define a new operator $\widetilde{L}$ taking

$$\widetilde{L} v = \frac{\partial}{\partial x_j} A_{ij}(x_0) \frac{\partial v}{\partial x_j} = A_{ij}(x_0) \frac{\partial^2 v}{\partial x_i \partial x_j}. \tag{4}$$

Note that $\widetilde{L}$ is also a uniformly elliptic operator with the same constants λ, Λ as L. Let v an $\widetilde{L}$ harmonic function with $v - u$ on the boundary of a ball $B_s(x_0)$, and define

$$\widetilde{A}_{x,r} = \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} v. \tag{5}$$

The inequality $(a+b)^2 \leq 2a^2 + 2b^2$ will be very useful in this proof. To apply Morrey we need to estimate $\int |\nabla u|^2$. First calculate

$$\int_{B_r(x_0)} (u - A_{x_0,r})^2 \leq 2\int_{B_r(x_0)} (u-v)^2 + 2\int_{B_r(x_0)} (v - A_{x_0,r})^2 \quad (6)$$

$$\leq 2\int_{B_r(x_0)} (u-v)^2 + 4\int_{B_r(x_0)} (v - \widetilde{A}_{x_0,r})^2 + 4\int_{B_r(x_0)} (\widetilde{A}_{x_0,r} - A_{x_0,r})^2 \quad (7)$$

Work on the last term. We have

$$\int_{B_r(x_0)} (\widetilde{A}_{x_0,r} - A_{x_0,r})^2 = \frac{1}{\text{vol } B_r(x_0)} \left(\int_{B_r(x_0)} (v-u) \right)^2 \quad (8)$$

$$\leq \int_{B_r(x_0)} (v-u)^2 \quad (9)$$

by Cauchy-Scwarz. Therefore we can absorb this into the first term of (8) to get

$$\int_{B_r(x_0)} (u - A_{x_0,r})^2 \leq 6\int_{B_r(x_0)} (u-v)^2 + 4\int_{B_r(x_0)} (v - \widetilde{A}_{x_0,r})^2. \quad (10)$$

Pick $s \geq r$ and expand the area of integration to get

$$\int_{B_r(x_0)} (u - A_{x_0,r})^2 \leq 6\int_{B_s(x_0)} (u-v)^2 + 4\int_{B_s(x_0)} (v - \widetilde{A}_{x_0,r})^2. \quad (11)$$

Note that $\widetilde{L}$ is in the correct form for the inequalities we proved last time to apply, so we can use them to get

$$\int_{B_r(x_0)} (u - A_{x_0,r})^2 \leq 6\int_{B_r(x_0)} (u-v)^2 + k\left(\frac{r}{s}\right)^{n+2} \int_{B_s(x_0)} (v - \widetilde{A}_{x_0,r})^2. \quad (12)$$

Similarly we can show

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \leq 6\int_{B_r(x_0)} |\nabla(u-v)|^2 + k'\left(\frac{r}{s}\right)^{n+2} \int_{B_s(x_0)} |\nabla v - \widetilde{(\nabla v)}_{x_0,r}|^2. \quad (13)$$

Now we need to estimate $\int |\nabla(u-v)|^2$. We'll prove a lemma that will be useful next time.

Lemma 1.2 *Let* $||A_{ij} - A_{ij}(x_0)|| = \sup_{B_s(x_0)} |A_{ij} - A_{ij}(x_0)|$. *Then*

$$\int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \left(\frac{||A_{ij} - A_{ij}(x_0)||}{\lambda}\right)^2 \int_{B_s(x_0)} |\nabla v|^2 \quad (14)$$

and

$$\int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \left(\frac{||A_{ij} - A_{ij}(x_0)||}{\lambda}\right)^2 \int_{B_s(x_0)} |\nabla u|^2 \quad (15)$$

We will prove only the first of these statements. The proof of the other is very similar.

Proof Calculate

$$\lambda \int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \int_{B_s(x_0)} A_{ij} \frac{\partial(v-u)}{\partial x_i} \frac{\partial(u-v)}{\partial x_j} \tag{16}$$

$$\leq \int_{B_s(x_0)} A_{ij} \frac{\partial(v-u)}{\partial x_i} \frac{\partial u}{\partial x_j} - \int_{B_s(x_0)} A_{ij} \frac{\partial(v-u)}{\partial x_i} \frac{\partial v}{\partial x_j}. \tag{17}$$

Work on the first term. Clearly $\int_{\partial B_s(x_0)} (u-v) A \nabla u \cdot dS = 0$. By Stokes' theorem we get

$$\int_{B_s(x_0)} A_{ij} \frac{\partial(v-u)}{\partial x_i} \frac{\partial u}{\partial x_j} = -\int_{B_s(x_0)} (u-v) \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j} = \int_{B_s(x_0)} (u-v) L u = 0. \tag{18}$$

Plugging this into 17 gives

$$\lambda \int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \int_{B_s(x_0)} A_{ij} \frac{\partial(v-u)}{\partial x_i} \frac{\partial v}{\partial x_j}. \tag{19}$$

By a similar calculation to 18 we get $\int_{B_s(x_0)} A_{ij}(x_0) \frac{\partial(v-u)}{\partial x_i} \frac{\partial v}{\partial x_j} = 0$, and

$$\lambda \int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \int_{B_s(x_0)} (A_{ij} - A_{ij}(x_0)) \frac{\partial(v-u)}{\partial x_i} \frac{\partial v}{\partial x_j} \tag{20}$$

$$\leq ||A_{ij} - A_{ij}(x_0)|| \int_{B_s(x_0)} \left| \frac{\partial(v-u)}{\partial x_i} \frac{\partial v}{\partial x_j} \right| \tag{21}$$

$$\leq ||A_{ij} - A_{ij}(x_0)|| \left(\int_{B_s(x_0)} |\nabla(u-v)|^2 \right)^{1/2} \left(\int_{B_s(x_0)} |\nabla v|^2 \right)^{1/2}) \tag{22}$$

Finally divide and square to get

$$\int_{B_s(x_0)} |\nabla(u-v)|^2 \leq \left(\frac{||A_{ij} - A_{ij}(x_0)||}{\lambda} \right)^2 \int_{B_s(x_0)} |\nabla v|^2 \tag{23}$$

as required. ∎