Lecture Notes for LG's Diff. Analysis

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1 The Sobolev Inequality

Suppose that $u \in C_c^1(\mathbb{R}^n)$. Clearly, if $\nabla u = 0$, then $u = 0$. So, we can ask if ∇u is "small", does this imply that u is small?

Question 1. If $u \in C_c^1(\mathbb{R}^n)$ and $\int |\nabla u| = 1$, is there a bound for $\sup |u|$?

Answer 1. If $n = 1$, then this is true by the Fundamental theorem of calculus.

If $n > 1$, then we have the following scaling example: Let $\eta \in C_c^1$ be a fixed smooth bump function. Define $\eta_\lambda(x) = \eta(x/\lambda)$. Then $\sup(\eta_\lambda) = \sup(\eta)$, and

$$\int |\nabla \eta_\lambda| dx = \lambda^{-1} \int |(\nabla \eta)(x/\lambda)| dx = \lambda^{n-1} \int |\nabla \eta| = \lambda^{n-1}$$

Therefore, we can make the L^1 norm of the derivative as small as we like, while keeping the L^∞ norm of the function large.

Theorem 1.1 (Sobolev Inequality)**.** If $u \in C_c^1(\mathbb{R}^n)$ then $\|u\|_{L^{\frac{n}{n-1}}} \leq \|\nabla u\|_{L^1}$.

Remark 1.1. Note that this will not hold true for $p \neq \frac{n}{n-1}$. To see this, suppose that we have $\|u\|_{L^p} \leq \|\nabla u\|_{L^1}$. As in the scaling example, pick η a smooth bump function, and define η_λ as before. Then

$$
\begin{aligned}
\int |\eta_\lambda|^p = \lambda^n \int \eta^p dx &\le \lambda^n \left(\int |\nabla \eta| dx \right)^p \\
&= \lambda^n \left(\lambda^{-n} \int |(\nabla \eta)(x/\lambda)| dx \right)^p \\
&= \lambda^n \left(\lambda^{-n+1} \int |\nabla \eta_\lambda| dx \right)^p \\
&= \lambda^{n+(1-n)p} \left(\int |\nabla \eta_\lambda| dx \right)^p
\end{aligned}
$$

Thus, if $p \neq n/(n-1)$, we can make the right hand side very small simply by making λ either large or small.

Before we prove the Sobolev Inequality, we'll prove a slightly easier problem:

Lemma 1.1. Let $u \in C_c^1(\mathbb{R}^n)$, $U = \{|u| > 1\}$ and $\pi_j : \mathbb{R}^n \to x_j^\perp$. Then $Vol_{n-1}(\pi_j(U)) \le \int |\nabla u|$.

Proof. WLOG, assume $j = n$. Then

$$
\begin{aligned}
Vol(\pi_j(U)) &\le \int_{\mathbb{R}^{n-1}} \max_{x_n} |u(x_1, \cdots, x_n)| dx_1 \cdots dx_{n-1} \\
&\le \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |\partial_n u(x_1, \cdots, x_n)| dx_n dx_1 \cdots dx_{n-1} \\
&\le \int_{\mathbb{R}^n} |\nabla u|
\end{aligned}
$$

□

Then we can use the Loomis-Whitney theorem which we proved in the homework:

Theorem 1.2 (Loomis-Whitney)**.** If $U \subset \mathbb{R}^n$ is open, and $|\pi_j(U)| \le A$ for all j, then $|U| \le A^{n/(n-1)}$.

Proof of Sobolev dimension 2. Define

$$u_1(x_2) = \int |\partial_1 u(x_1, x_2)| dx_1$$
$$u_2(x_1) = \int |\partial_2 u(x_1, x_2)| dx_2$$

Then $|u(x_1, x_2)| \leq u_i(x_j)$. Therefore,

$$\int u^2 \leq \int u_1(x_2) u_2(x_1) dx_1 dx_2 = \left(\int u_1 dx_2 \right) \left(\int u_2 dx_1 \right) \leq \left(\int |\nabla u| \right)^2$$

□

With this in hand, we can move on to the proof of the Sobolev by induction.

Proof of General Sobolev. Define

$$u_n(x_1, \cdots, x_{n-1}) = \int |\partial_n u(x_1, \cdots, x_n)| dx_n$$

Then $|u| \leq u_n$. and $\int_{\mathbb{R}^{n-1}} u_n \leq \int_{\mathbb{R}^n} |\nabla u|$.

We will proceed by induction.

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^{n/(n-1)} &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} |u| u_n^{1/(n-1)} dx_1 \cdots dx_{n-1} \right) dx_n \\
&\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |u|^{\frac{n-1}{n-2}} \right]^{\frac{n-2}{n-1}} \left[\int_{\mathbb{R}^{n-1}} |u_n| \right]^{1/(n-1)} dx_n \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\nabla u| dx_1 \cdots dx_{n-1} dx_n \left(\int_{\mathbb{R}^{n-1}} u_n \right)^{1/(n-1)} \\
&\leq \left(\int |\nabla u| \right)^{n/(n-1)}
\end{aligned}$$