Lecture 6

Previously, we started with the continuous PDE equations and derived a discrete/matrix version as an approximation. Now we will do the reverse: we will *start* with a truly discrete (finite-dimensional) system, and then derive the continuum PDE model as a limit or approximation. This is one of the common ways by which PDE models are derived in the first place, will give yet another perspective on similar mathematics, and will also shed some light on the origin of the variable c(x) coefficients from the last lecture.

In particular, we look at the 8.01 (or 8.03) system of a set of N masses m sliding without friction and connected by springs of constant k, anchored to walls on both sides.

If the displacements of each mass are given by a vector $\mathbf{u}$, we show that $d^2\mathbf{u}/dt^2 = A\mathbf{u}$ where $A = (k/m) (-D^T D^T) \Delta x^2$, with D being *exactly* our "first derivative" matrix from previous lectures. Thus, up to a scale factor, our discrete Laplacian A from previous lectures is not only an approximation for d^2/dx^2, but it is also *exactly* the matrix describing this coupled-spring system!

By inspection, A is real-symmetric negative-definite as before, and hence we have N real eigenvalues $\lambda_n < 0$ and orthonormal eigenvectors $\mathbf{u}_n$. By expanding the solution $\mathbf{u}(t)$ in terms of these eigenvectors, we see that we obtain oscillating solutions: a set of (orthogonal) *normal modes* with real "eigenfrequencies" $\omega_n = \sqrt{-\lambda_n}$. The negative-definiteness is critical to have oscillation, as otherwise we would get complex ω_n and exponential growth! Showed a few examples for N=1,2,3 to get an intuition for these modes.

Took the N→∞ limit keeping the total length L fixed, which corresponds to $\Delta x \to 0$ (breaking the system into smaller and smaller pieces). Correspondingly, we decrease the masses proportional to Δx, so that $m = \rho \Delta x$ where ρ is a density. On the other hand, reminded ourselves that cutting springs in half *increases* the spring constant, so we should let $k = c/\Delta x$ for some constant c. With these definitions, our matrix A is precisely $(c/\rho) (-D^T D)$, where $-D^T D$ is our approximate (center-difference) d^2/dx^2 from before, and hence this limit gives precisely the scalar wave equation $\partial^2 u/\partial t^2 = (c/\rho) \partial^2 u/\partial x^2$, with the boundary conditions $u(0,t) = u(L,t) = 0$ (fixed ends). As before, this operator is self-adjoint negative definite and so we get real $\lambda_n < 0$ etcetera. Exactly as in the finite-dimensional case above, we therefore get oscillating "normal mode" solutions, just with an infinite series of eigenfunctions instead of a finite sum.

Finally, considered the "inhomogeneous medium" case where we allow all the masses and spring constants to be different. Showed that this corresponds to inserting some diagonal matrices into the above expressions, and in the continuum limit gives $\partial^2 u/\partial t^2 = (1/\rho) \partial/\partial x (c \, \partial u/\partial x) = \hat{A}u$ where $\rho(x)$ and $c(x)$ are positive functions. As in the previous lecture, we can see that this $\hat{A}$ is indeed self-adjoint negative-definite, and hence we get *oscillating normal-mode solutions*, if we define the modified inner product $\langle u,v \rangle = \int \rho u v$. And we can see this even though we probably cannot solve this PDE by hand except for very special cases of ρ and c!

Further reading: Same as previous lecture. Sections 2.1 and 2.2 in the Strang book cover very similar material on discrete vibration problems.

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