Solutions to Practice Test 1

18.303 Linear Partial Differential Equations

Matthew J. Hancock

Fall 2006

1 Given

You may assume the eigenvalues of the Sturm-Liouville problem

$$\begin{aligned} X'' + \lambda X &= 0, \qquad 0 < x < 1 \\ X(0) &= 0 \qquad X(1) = 0 \end{aligned}$$

are $\lambda_n = n^2\pi^2$ and $X_n(x) = \sin(nx)$, for $n = 1, 2, ...$, without derivation.

You may also assume the following orthogonality conditions for m, n positive integers:

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

2 Question

Consider the following heat problem in dimensionless variables

$$u_t = u_{xx} + \frac{\pi^2}{4}u - b, \qquad 0 < x < 1, \qquad t > 0 \tag{1}$$

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad t > 0 \tag{2}$$

$$u(x,0) = u_0 \qquad 0 < x < 1. \tag{3}$$

(a) [3 points] Explain in terms of a heated rod precisely what the problem models mathematically.

Solution: The problem models heat transfer in a rod of (scaled) length 1, with thermal diffusivity 1. The temperature is fixed at zero degrees at both ends and the rod is initially at a constant temperature u_0. Heat is absorbed througout the rod at a rate of b and produced/absorbed at a rate proportional to the current temperature (proportionality constant 1/4).

(b) [3 points] Derive the equilibrium solution

$$u_E(x) = \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right)$$

It is insufficient to simply verify that the solution works.

Solution: The equilibrium solution $u_E(x)$ satisfies

$$u_E''(x) + \frac{\pi^2}{4} u_E(x) = b$$

$$u_E(0) = 0 = u_E(1)$$

The ODE has solution

$$u_E(x) = A\cos\left(\frac{\pi x}{2}\right) + B\sin\left(\frac{\pi x}{2}\right) + \frac{4b}{\pi^2}$$

Imposing the BCs gives

$$\begin{aligned} u_E(0) &= A + 4b/\pi^2 = 0 \\ u_E(1) &= B + 4b/\pi^2 = 0 \end{aligned}$$

Solving for A, B gives $A = B = -4b/\pi^2$. Putting things together gives

$$u_E(x) = \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right)$$

(c) [3 points] Using $u_E(x)$, transform the given heat problem for $u(x,t)$ into the following problem for a function $v(x,t)$:

$$\begin{aligned} v_t &= v_{xx} + \frac{\pi^2}{4} v, \qquad 0 < x < 1, \qquad t > 0 &(4) \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0 &(5) \\ v(x,0) &= f(x) \qquad 0 < x < 1. &(6) \end{aligned}$$

where $f(x)$ will be determined by the transformation.

Solution: We let

$$v(x,t) = u(x,t) - u_E(x)$$

or

$$u(x,t) = v(x,t) + u_E(x)$$

Then

$$u_t = v_t, \qquad u_{xx} = v_{xx} + u_E'' = v_{xx} + b\left(\cos\left(\frac{\pi x}{2}\right) + \sin\left(\frac{\pi x}{2}\right)\right)$$

so that the PDE (1) for $u(x,t)$ becomes

$$\begin{aligned} v_t &= v_{xx} + b\left(\cos\left(\frac{\pi x}{2}\right) + \sin\left(\frac{\pi x}{2}\right)\right) + \frac{\pi^2}{4}u_E + \frac{\pi^2}{4}v - b \\ &= v_{xx} + \frac{\pi^2}{4}v \end{aligned}$$

Thus, the PDE becomes

$$v_t = v_{xx} + \frac{\pi^2}{4}v$$

The BCs (2) become

$$\begin{aligned} v(0,t) &= u(0,t) - u_E(0) = 0 - 0 = 0 \\ v(1,t) &= u(1,t) - u_E(1) = 0 - 0 = 0 \end{aligned}$$

The IC (3) becomes

$$v(x,0) = u(x,0) - u_E(x) = u_0 - \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right)$$

We have shown that $v(x,t)$ satisfies the PDE (4), BCs (5) and the IC (6) with

$$f(x) = u_0 - \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right) \tag{7}$$

(d) [3 points] For an appropriate value of α show that the transformation $w(x,t) = e^{\alpha t}v(x,t)$ further simplifies the problem to

$$\begin{aligned} w_t &= w_{xx}, \qquad 0 < x < 1, \qquad t > 0 && (8) \\ w(0,t) &= 0, \qquad w(1,t) = 0, \qquad t > 0 && (9) \\ w(x,0) &= f(x) \qquad 0 < x < 1. && (10) \end{aligned}$$

Solution: Letting $w(x,t) = e^{\alpha t}v(x,t)$, the BCs (5) and IC (6) become

$$\begin{aligned} w(0,t) &= e^{\alpha t}v(0,t) = 0, \\ w(1,t) &= e^{\alpha t}v(1,t) = 0, \end{aligned}$$

$$w(x,0) = v(x,0) = f(x)$$

To transform the PDE, note that $v(x,t) = e^{-\alpha t} w(x,t)$ and hence

$$\begin{aligned} v_t &= -\alpha e^{-\alpha t} w + e^{-\alpha t} w_t \\ v_{xx} &= e^{-\alpha t} w_{xx} \end{aligned}$$

so the PDE (4) for $v(x,t)$ becomes

$$-\alpha e^{-\alpha t} w + e^{-\alpha t} w_t = e^{-\alpha t} w_{xx} + \frac{\pi^2}{4} e^{-\alpha t} w$$

Multiplying by $e^{\alpha t}$ and rearranging gives

$$w_t = w_{xx} + \left(\alpha + \frac{\pi^2}{4}\right) w$$

Choosing $\alpha = -\pi^2/4$ yields

$$w_t = w_{xx}$$

with $v(x,t) = e^{\pi^2 t/4} w(x,t)$. We have shown that $w(x,t)$ satisfies the PDE (8), BCs (9) and the IC (10) with $f(x)$ given in (7).

(e) [8 points] Derive the solution

$$w(x,t) = \sum_{n=1}^{\infty} w_n(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{2(u_0 - 4b/\pi^2)}{2n-1} + \frac{32b(2n-1)}{\pi^2 (4n-3)(4n-1)} \right) e^{-(2n-1)^2 \pi^2 t} \sin((2n-1)\pi x)$$

and hence solve for $u(x,t) = u_E(x) + \sum_{n=1}^{\infty} u_n(x,t)$ using the earlier transformations.

Solution: Note that the PDE (8), BCs (9) and the IC (10) are the basic heat problem we considered in class. We derived the solution using separation of variables,

$$w(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2\pi^2 t} \tag{11}$$

where

$$B_n = 2\int_0^1 w(x,0)\sin(n\pi x)\,dx = 2\int_0^1 f(x)\sin(n\pi x)\,dx \tag{12}$$

and $f(x)$ is given in (7). Note that

$$\begin{aligned} \int_0^1 \sin(n\pi x)\,dx &= \frac{1}{n\pi}\left[-\cos(n\pi x)\right]_0^1 \\ &= \frac{1}{n\pi}(1-\cos(n\pi)) = \frac{1}{n\pi}\left(1-(-1)^n\right) \end{aligned}$$

$$\begin{aligned}
\int_0^1 \cos\left(\frac{\pi x}{2}\right)\sin(n\pi x)\,dx &= \int_0^1 \frac{1}{2}\left(\sin\left(\frac{2n+1}{2}\pi x\right)+\sin\left(\frac{2n-1}{2}\pi x\right)\right)dx \\
&= \frac{1}{2}\left[-\frac{2\cos\left(\frac{2n+1}{2}\pi x\right)}{(2n+1)\pi}-\frac{2\cos\left(\frac{2n-1}{2}\pi x\right)}{(2n-1)\pi}\right]_0^1 \\
&= \frac{1}{(2n+1)\pi}+\frac{1}{(2n-1)\pi} \\
&= \frac{4n}{(2n+1)(2n-1)\pi}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \sin\left(\frac{\pi x}{2}\right)\sin(n\pi x)\,dx &= \int_0^1 \frac{1}{2}\left(-\cos\left(\frac{2n+1}{2}\pi x\right)+\cos\left(\frac{2n-1}{2}\pi x\right)\right)dx \\
&= \frac{1}{2}\left[-\frac{2\sin\left(\frac{2n+1}{2}\pi x\right)}{(2n+1)\pi}+\frac{2\sin\left(\frac{2n-1}{2}\pi x\right)}{(2n-1)\pi}\right]_0^1 \\
&= -\frac{\sin\left(\frac{2n+1}{2}\pi\right)}{(2n+1)\pi}+\frac{\sin\left(\frac{2n-1}{2}\pi\right)}{(2n-1)\pi} \\
&= -\frac{(-1)^n}{(2n+1)\pi}+\frac{(-1)^{n+1}}{(2n-1)\pi} \\
&= -\frac{4n(-1)^n}{(2n+1)(2n-1)\pi}
\end{aligned}$$

Thus (12) becomes

$$\begin{aligned}
B_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx \\
&= 2\int_0^1 \left(u_0-\frac{4b}{\pi^2}\left(1-\cos\left(\frac{\pi x}{2}\right)-\sin\left(\frac{\pi x}{2}\right)\right)\right)\sin(n\pi x)\,dx \\
&= 2\left(u_0-\frac{4b}{\pi^2}\right)\int_0^1 \sin(n\pi x)\,dx \\
&\quad +\frac{8b}{\pi^2}\int_0^1\left(\cos\left(\frac{\pi x}{2}\right)+\sin\left(\frac{\pi x}{2}\right)\right)\sin(n\pi x)\,dx \\
&= \frac{2}{n\pi}\left(u_0-\frac{4b}{\pi^2}\right)(1-(-1)^n)+\frac{16bn\,(1-(-1)^n)}{\pi^3(2n+1)(2n-1)} \\
&= \begin{cases} \frac{4\left(u_0-4b/\pi^2\right)}{(2m-1)\pi}+\frac{32b(2m-1)}{(4m-1)(4m-3)\pi^2}, & n=2m-1 \text{ odd} \\ 0 & n \text{ even}\end{cases}
\end{aligned}$$

Substituting B_n into (11) gives

$$w(x,t)=\sum_{m=1}^{\infty}\frac{2}{\pi}\left(\frac{2\left(u_0-4b/\pi^2\right)}{2m-1}+\frac{32b\,(2m-1)}{\pi^2(4m-1)(4m-3)}\right)\sin((2m-1)\pi x)\,e^{-(2m-1)^2\pi^2 t}$$

as required. The solution $u(x,t)$ is given by reversing our transformations,

$$\begin{aligned} u(x,t) &= e^{\pi^2 t/4} w(x,t) + u_E(x) \\ &= e^{\pi^2 t/4} \sum_{m=1}^{\infty} \frac{2}{\pi} \left(\frac{2(u_0 - 4b/\pi^2)}{2m-1} + \frac{32b(2m-1)}{\pi^2 (4m-1)(4m-3)} \right) \sin((2m-1)\pi x) e^{-(2m-1)^2 \pi^2 t} \\ &\quad + \frac{4b}{\pi^2} \left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right) \right) \end{aligned}$$

Aside (optional): a quick check of the above formula for $w(x,t)$:

1. $w(0,t) = 0 = w(1,t)$
2. $w(x,0) =$ fourier series of $f(x)$
3. $w_t = w_{xx}$ since $\sin((2m-1)\pi x) e^{-(2m-1)^2\pi^2 t}$ satisfies the PDE for all m.

(f) [4 points] Prove that the solution $u(x,t)$ is unique. [Hint: first show that $w(x,t)$ is unique].

Solution: We follow the standard uniqueness proof we used in class and on the assignments. Suppose w_1 and w_2 both satisfy the PDE (8), BCs (9) and the IC (10). Then $h(x,t) = w_1(x,t) - w_2(x,t)$ satisfies

$$\begin{aligned} h_t &= h_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\ h(0,t) &= 0, \qquad h(1,t) = 0, \qquad t > 0 \\ h(x,0) &= 0 \qquad 0 < x < 1. \end{aligned}$$

Define

$$H(t) = \int_0^1 h^2(x,t)\, dx$$

Differentiate in time,

$$\begin{aligned} \frac{dH}{dt} &= \int_0^1 2hh_t dx = \int_0^1 2hh_{xx} dx, \qquad \text{by PDE} \\ &= 2[hh_x]_0^1 - 2\int_0^1 h_x^2 dx, \qquad \text{integrating by parts} \\ &= -2\int_0^1 h_x^2 dx. \qquad \text{applying the BCs} \end{aligned}$$

Thus $dH/dt \leq 0$. Now $H(t) \geq 0$ since the integrand is everywhere non-negative. Also, $H(0) = 0$ since $h(x,0) = 0$ for all x. Thus $H(t)$ is a non-negative non-increasing function that starts at 0, and hence $H(t)$ must be zero for all time t. This implies, since the integrand $h(x,t)$ is non-negative, that $h(x,t) = 0$ for all t and x. Hence $w_1(x,t) = w_2(x,t)$ and the solution $w(x,t)$ is unique.

Since $u(x,t)$ is obtained from $w(x,t)$ by the one-to-one transformation

$$u(x,t) = e^{\pi^2 t/4} w(x,t) + u_E(x)$$

then the solution $u(x,t)$ is also unique.

(g) [6 points] Let $u_0 = 4b/\pi^2$. Show that

$$\left|\frac{u_2(x,t)}{u_1(x,t)}\right| \leq \frac{27}{35} e^{-8}, \qquad t \geq 1/\pi^2.$$

Hence show that

$$u(x,t) \approx u_E(x) + A_1 e^{-3\pi^2 t/4} \sin(\pi x)$$

is a good approximation for $t \geq 1/\pi^2$. Sketch $u = u_0$ and $u = u_E(x)$ for $0 < x < 1$ and comment on the physical significance of the sign of A_1.

Solution: When $u_0 = 4b/\pi^2$, the solution $u(x,t)$ becomes

$$\begin{aligned} u(x,t) &= \sum_{m=1}^{\infty} u_m(x,t) + \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right) \\ &= \sum_{m=1}^{\infty} \frac{64b(2m-1)}{\pi^3(4m-1)(4m-3)} \sin((2m-1)\pi x)\, e^{-\left[(2m-1)^2\pi^2 - \pi^2/4\right]t} \\ &\quad + \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right) \end{aligned}$$

where

$$u_m(x,t) = \frac{64b(2m-1)}{\pi^3(4m-1)(4m-3)} \sin((2m-1)\pi x)\, e^{-\left[(2m-1)^2\pi^2 - \pi^2/4\right]t}$$

Thus

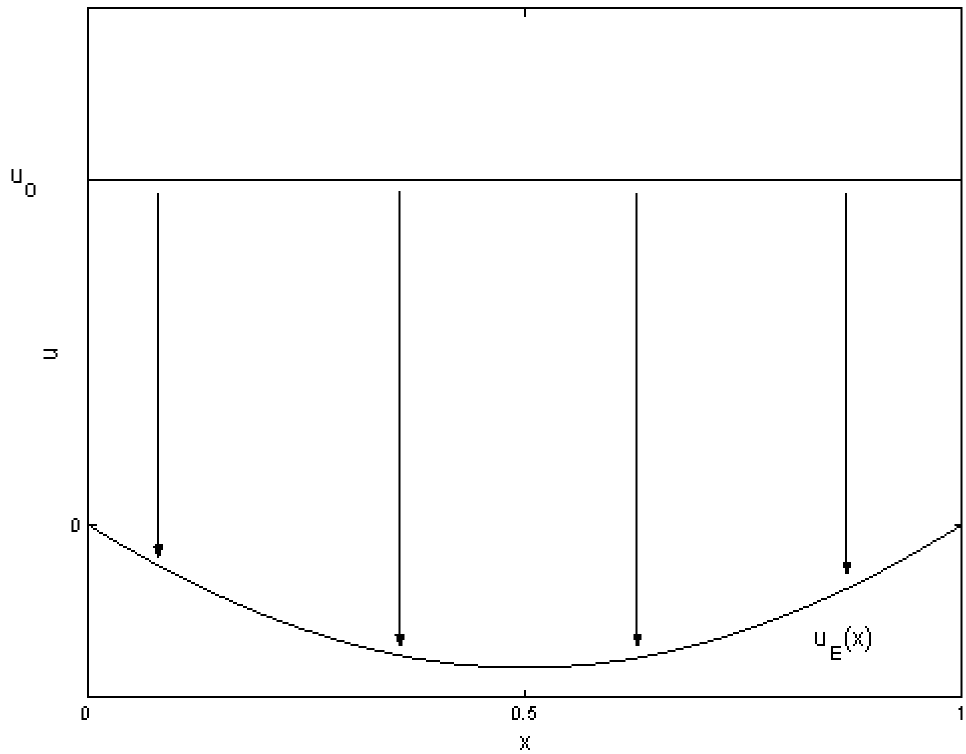
$$\begin{aligned} u_1(x,t) &= \frac{64b}{3\pi^3} \sin(\pi x)\, e^{-\left[\pi^2 - \pi^2/4\right]t} \\ u_2(x,t) &= \frac{192b}{35\pi^3} \sin(3\pi x)\, e^{-\left[9\pi^2 - \pi^2/4\right]t} \end{aligned}$$

Thus

$$\left|\frac{u_2(x,t)}{u_1(x,t)}\right| = \left|\frac{\frac{192b}{35\pi^3}\sin(3\pi x)\, e^{-\left[9\pi^2 - \pi^2/4\right]t}}{\frac{64b}{3\pi^3}\sin(\pi x)\, e^{-\left[\pi^2 - \pi^2/4\right]t}}\right| = \frac{9}{35} e^{-8\pi^2 t} \left|\frac{\sin(3\pi x)}{\sin(\pi x)}\right|$$

and using $|\sin n\pi x| \leq n |\sin \pi x|$ we have

$$\left|\frac{u_2(x,t)}{u_1(x,t)}\right| \leq \frac{27}{35} e^{-8\pi^2 t} \leq e^{-8}, \qquad t \geq 1/\pi^2.$$



Thus, the first term dominates the others for $t \geq 1/\pi^2$, so that

$$\begin{aligned} u(x,t) &\approx u_E(x) + u_1(x,t) \\ &= \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right) + \frac{64b}{3\pi^3}\sin(\pi x)\, e^{-3\pi^2 t/4} \end{aligned}$$

is a good approximation for $t \geq 1/\pi^2$. Thus $A_1 = 64b/(3\pi^3)$.

The sketch of $u = u_0 > 0$ and $u = u_E(x)$ for $0 < x < 1$ is shown below. We assume that b is a source so that $b > 0$ and $A_1 > 0$. Thus, the rod cools down to the equilibrium everywhere.