The heat and wave equations in 2D and 3D

18.303 Linear Partial Differential Equations

Matthew J. Hancock

Fall 2006

1 2D and 3D Heat Equation

Ref: Myint-U & Debnath §2.3 – 2.5

[Nov 2, 2006]

Consider an arbitrary 3D subregion V of $\mathbb{R}^3$ ($V \subseteq \mathbb{R}^3$), with temperature $u(\mathbf{x}, t)$ defined at all points $\mathbf{x} = (x, y, z) \in V$. We generalize the ideas of 1-D heat flux to find an equation governing u. The heat energy in the subregion is defined as

$$\text{heat energy } = \int\int_V c\rho u \, dV$$

Recall that conservation of energy implies

$$\begin{matrix}\text{rate of change} \\ \text{of heat energy}\end{matrix} = \begin{matrix}\text{heat energy into } V \text{ from} \\ \text{boundaries per unit time}\end{matrix} + \begin{matrix}\text{heat energy generated} \\ \text{in solid per unit time}\end{matrix}$$

We desire the heat flux through the boundary S of the subregion V, which is the normal component of the heat flux vector ϕ, $\phi \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward unit normal at the boundary S. Hats on vectors denote a unit vector, $|\hat{\mathbf{n}}| = 1$ (length 1). If the heat flux vector ϕ is directed inward, then $\phi \cdot \hat{\mathbf{n}} < 0$ and the outward flow of heat is negative. To compute the total heat energy flowing across the boundaries, we sum $\phi \cdot \hat{\mathbf{n}}$ over the entire closed surface S, denoted by a double integral $\int\int_S dS$. Therefore, the conservation of energy principle becomes

$$\frac{d}{dt}\int\int\int_V c\rho u \, dV = -\int\int_S \phi \cdot \hat{\mathbf{n}} \, dS + \int\int\int_V Q \, dV \tag{1}$$

1.1 Divergence Theorem (a.k.a. Gauss's Theorem)

For any volume V with closed smooth surface S,

$$\int\int\int_V \nabla \cdot \mathbf{A}\, dV = \int\int_S \mathbf{A} \cdot \hat{\mathbf{n}}\, dS$$

where $\mathbf{A}$ is any function that is smooth (i.e. continuously differentiable) for $\mathbf{x} \in V$. Note that

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

where $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_z$ are the unit vectors in the x, y, z directions, respectively. The divergence of a vector valued function $\mathbf{F} = (F_x, F_y, F_z)$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

The Laplacian of a scalar function F is

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}.$$

Applying the Divergence theorem to (1) gives

$$\frac{d}{dt}\int\int\int_V c\rho u\, dV = -\int\int\int_V \nabla \cdot \phi\, dV + \int\int\int_V Q\, dV$$

Since V is independent of time, the integrals can be combined as

$$\int\int\int_V \left(c\rho\frac{\partial u}{\partial t} + \nabla \cdot \phi - Q\right) dV = 0$$

Since V is an arbitrary subregion of $\mathbb{R}^3$ and the integrand is assumed continuous, the integrand must be everywhere zero,

$$c\rho\frac{\partial u}{\partial t} + \nabla \cdot \phi - Q = 0 \qquad (2)$$

1.2 Fourier's Law of Heat Conduction

The 3D generalization of Fourier's Law of Heat Conduction is

$$\phi = -K_0 \nabla u \qquad (3)$$

where K_0 is called the thermal diffusivity. Substituting (3) into (2) gives

$$\frac{\partial u}{\partial t} = \kappa\nabla^2 u + \frac{Q}{c\rho} \qquad (4)$$

where $\kappa = K_0 / (c\rho)$. This is the 3D Heat Equation. Normalizing as for the 1D case,

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{l}, \qquad \tilde{t} = \frac{\kappa}{l^2} t,$$

Eq. (4) becomes (dropping tildes) the non-dimensional Heat Equation,

$$\frac{\partial u}{\partial t} = \nabla^2 u + q, \tag{5}$$

where $q = l^2 Q / (\kappa c \rho) = l^2 Q / K_0$.

2 2D and 3D Wave equation

The 1D wave equation can be generalized to a 2D or 3D wave equation, in scaled coordinates,

$$u_{tt} = \nabla^2 u \tag{6}$$

This models vibrations on a 2D membrane, reflection and refraction of electromagnetic (light) and acoustic (sound) waves in air, fluid, or other medium.

3 Separation of variables in 2D and 3D

Ref: Guenther & Lee §10.2, Myint-U & Debnath §4.10, 4.11

We consider simple subregions $D \subseteq \mathbb{R}^3$. We assume the boundary conditions are zero, $u = 0$ on ∂D, where ∂D denotes the closed surface of D (assumed smooth). The 3D Heat Problem is

$$\begin{aligned} u_t &= \nabla^2 u, \qquad \mathbf{x} \in D, \qquad t > 0, \\ u(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \partial D, \\ u(\mathbf{x}, 0) &= f(\mathbf{x}), \qquad \mathbf{x} \in D. \end{aligned} \tag{7}$$

The 3D wave problem is

$$\begin{aligned} u_{tt} &= \nabla^2 u, \qquad \mathbf{x} \in D, \qquad t > 0, \\ u(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \partial D, \\ u(\mathbf{x}, 0) &= f(\mathbf{x}), \qquad \mathbf{x} \in D, \\ u_t(\mathbf{x}, 0) &= g(\mathbf{x}), \qquad \mathbf{x} \in D. \end{aligned} \tag{8}$$

We separate variables as

$$u(\mathbf{x}, t) = X(\mathbf{x}) T(t) \tag{9}$$

The 3D Heat Equation implies

$$\frac{T'}{T} = \frac{\nabla^2 X}{X} = -\lambda = const \tag{10}$$

where $\lambda = const$ since the l.h.s. depends solely on t and the middle X''/X depends solely on $\mathbf{x}$. The 3D wave equation becomes

$$\frac{T''}{T} = \frac{\nabla^2 X}{X} = -\lambda = const \tag{11}$$

On the boundaries,

$$X(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial D$$

The Sturm-Liouville Problem for $X(\mathbf{x})$ is

$$\begin{aligned} \nabla^2 X + \lambda X &= 0, \qquad \mathbf{x} \in D \\ X(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \partial D \end{aligned} \tag{12}$$

4 Solution for $T(t)$

Suppose that the Sturm-Liouville problem (12) has eigen-solution $X_n(\mathbf{x})$ and eigen-value λ_n, where $X_n(\mathbf{x})$ is non-trivial. Then for the 3D Heat Problem, the problem for $T(t)$ is, from (10),

$$\frac{T'}{T} = -\lambda \tag{13}$$

with solution

$$T_n(t) = c_n e^{-\lambda_n t} \tag{14}$$

and the corresponding solution to the PDE and BCs is

$$u_n(\mathbf{x}, t) = X_n(\mathbf{x}) T_n(t) = X_n(\mathbf{x}) c_n e^{-\lambda_n t}.$$

For the 3D Wave Problem, the problem for $T(t)$ is, from (11),

$$\frac{T''}{T} = -\lambda \tag{15}$$

with solution

$$T_n(t) = \alpha_n \cos\left(\sqrt{\lambda_n} t\right) + \beta_n \sin\left(\sqrt{\lambda_n} t\right) \tag{16}$$

and the corresponding normal mode is $u_n(\mathbf{x}, t) = X_n(\mathbf{x}) T_n(t)$.

5 Uniqueness of the 3D Heat Problem

Ref: Guenther & Lee §10.3

We now prove that the solution of the 3D Heat Problem

$$\begin{aligned} u_t &= \nabla^2 u, \qquad \mathbf{x} \in D \\ u(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \partial D \\ u(\mathbf{x}, 0) &= f(\mathbf{x}), \qquad \mathbf{x} \in D \end{aligned}$$

is unique. Let u_1, u_2 be two solutions. Define $v = u_1 - u_2$. Then v satisfies

$$\begin{aligned} v_t &= \nabla^2 v, \qquad \mathbf{x} \in D \\ v(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \partial D \\ v(\mathbf{x}, 0) &= 0, \qquad \mathbf{x} \in D \end{aligned}$$

Let

$$V(t) = \int\int\int_D v^2 dV \geq 0$$

$V(t) \geq 0$ since the integrand $v^2(\mathbf{x}, t) \geq 0$ for all $(\mathbf{x}, t)$. Differentiating in time gives

$$\frac{dV}{dt}(t) = \int\int\int_D 2 v v_t dV$$

Substituting for v_t from the PDE yields

$$\frac{dV}{dt}(t) = \int\int\int_D 2 v \nabla^2 v dV$$

By result (26) derived below,

$$\frac{dV}{dt}(t) = 2\int\int_{\partial D} v \nabla v \cdot \hat{\mathbf{n}} dS - 2\int\int\int_D |\nabla v|^2 dV \tag{17}$$

But on ∂D, $v = 0$, so that the first integral on the r.h.s. vanishes. Thus

$$\frac{dV}{dt}(t) = -2\int\int\int_D |\nabla v|^2 dV \leq 0 \tag{18}$$

Also, at $t = 0$,

$$V(0) = \int\int\int_D v^2(\mathbf{x}, 0) dV = 0$$

Thus $V(0) = 0$, $V(t) \geq 0$ and $dV/dt \leq 0$, i.e. $V(t)$ is a non-negative, non-increasing function that starts at zero. Thus $V(t)$ must be zero for all time t, so that $v(\mathbf{x}, t)$ must be identically zero throughout the volume D for all time, implying the two solutions are the same, $u_1 = u_2$. Thus the solution to the 3D heat problem is unique.

For insulated BCs, $\nabla v = 0$ on ∂D, and hence $v \nabla v \cdot \hat{\mathbf{n}} = 0$ on ∂D. Thus we can still derive Eq. (18) from (17), and the uniqueness proof still holds. Thus the 3D Heat Problem with Type II homogeneous BCs also has a unique solution.

6 Sturm-Liouville problem

Ref: Guenther & Lee §10.2, Myint-U & Debnath §7.1 – 7.3

Both the 3D Heat Equation and the 3D Wave Equation lead to the Sturm-Liouville problem

$$\begin{aligned} \nabla^2 X + \lambda X &= 0, \qquad \mathbf{x} \in D, \\ X(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \partial D. \end{aligned} \tag{19}$$

6.1 Green's Formula and the Solvability Condition

Ref: Guenther & Lee §8.3, Myint-U & Debnath §10.10 (Exercise 1)

For the Type I BCs assumed here ($u(\mathbf{x}, t) = 0$, for $\mathbf{x} \in \partial D$), we now show that all eigenvalues are positive. To do so, we need a result that combines some vector calculus with the Divergence Theorem. From vector calculus, for any scalar function G and vector valued function $\mathbf{F}$,

$$\nabla \cdot (G\mathbf{F}) = G\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla G \tag{20}$$

Using the divergence theorem,

$$\int\int_S (G\mathbf{F}) \cdot \hat{\mathbf{n}} dS = \int\int\int_V \nabla \cdot (G\mathbf{F}) \, dV \tag{21}$$

Substituting (20) into (21) gives

$$\int\int_S (G\mathbf{F}) \cdot \hat{\mathbf{n}} dS = \int\int\int_V G\nabla \cdot \mathbf{F} dV + \int\int\int_V \mathbf{F} \cdot \nabla G dV \tag{22}$$

Result (22) applied to $G = v_1$ and $\mathbf{F} = \nabla v_2$ gives

$$\int\int_S (v_1 \nabla v_2) \cdot \hat{\mathbf{n}} dS = \int\int\int_V \left(v_1 \nabla^2 v_2 + \nabla v_2 \cdot \nabla v_1\right) dV \tag{23}$$

Result (22) applied to $G = v_2$ and $\mathbf{F} = \nabla v_1$ gives

$$\int\int_S (v_2 \nabla v_1) \cdot \hat{\mathbf{n}} dS = \int\int\int_V \left(v_2 \nabla^2 v_1 + \nabla v_1 \cdot \nabla v_2\right) dV \tag{24}$$

Subtracting (23) and (24) gives Green's Formula (also known as Green's second identity):

$$\int\int_S (v_1 \nabla v_2 - v_2 \nabla v_1) \cdot \hat{\mathbf{n}} dS = \int\int\int_V \left(v_1 \nabla^2 v_2 - v_2 \nabla^2 v_1\right) dV \tag{25}$$

This is also known as a Solvability Condition, since the values of v_1 and v_2 on the boundary of D must be consistent with the values of v_1 and v_2 on the interior of D. Result (25) holds for any smooth function v defined on a volume V with closed smooth surface S.

6.2 Positive, real eigenvalues (for Type I BCs)

Ref: Myint-U & Debnath §7.2

Choosing $G = v$ and $\mathbf{F} = \nabla v$, for some function v, in (22) gives

$$\begin{aligned} \int\int_S v\nabla v \cdot \hat{\mathbf{n}} dS &= \int\int\int_V \left(v\nabla^2 v + \nabla v \cdot \nabla v\right) dV \\ &= \int\int\int_V \left(v\nabla^2 v + |\nabla v|^2\right) dV. \end{aligned} \tag{26}$$

Result (26) holds for any smooth function v defined on a volume V with closed smooth surface S.

We now apply result (26) to a solution $X(\mathbf{x})$ of the Sturm-Liouville problem (19). Letting $v = X(\mathbf{x})$, $S = \partial D$ and $V = D$, Eq. (26) becomes

$$\int\int_{\partial D} X\nabla X \cdot \hat{\mathbf{n}} dS = \int\int\int_D X\nabla^2 X dV + \int\int\int_D |\nabla X|^2 dV \tag{27}$$

Since $X(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial D$,

$$\int\int_{\partial D} X\nabla X \cdot \hat{\mathbf{n}} dS = 0 \tag{28}$$

Also, from the PDE in (19),

$$\int\int\int_D X\nabla^2 X dV = -\lambda \int\int\int_D X^2 dV \tag{29}$$

Substituting (28) and (29) into (27) gives

$$0 = -\lambda \int\int\int_D X^2 dV + \int\int\int_D |\nabla X|^2 dV \tag{30}$$

For non-trivial solutions, $X \neq 0$ at some points in D and hence by continuity of X, $\int\int\int_D X^2 dV > 0$. Thus (30) can be rearranged,

$$\lambda = \frac{\int\int\int_D |\nabla X|^2 dV}{\int\int\int_D X^2 dV} \geq 0 \tag{31}$$

Since X is real, the the eigenvalue λ is also real.

If $\nabla X = 0$ for all points in D, then integrating and imposing the BC $X(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial D$ gives $X = 0$ for all $\mathbf{x} \in D$, i.e. the trivial solution. Thus ∇X is nonzero at some points in D, and hence by continuity of ∇X, $\int\int\int_D |\nabla X|^2 dV > 0$. Thus, from (31), $\lambda > 0$.

6.3 Orthogonality of eigen-solutions to Sturm-Liouville problem

Ref: Myint-U & Debnath §7.2

Suppose v_1, v_2 are two eigenfunctions with eigenvalues λ_1, λ_2 of the 3D Sturm-Liouville problem

$$\begin{aligned} \nabla^2 v + \lambda v &= 0, \qquad \mathbf{x} \in D \\ v &= 0, \qquad \mathbf{x} \in \partial D \end{aligned}$$

We make use of Green's Formula (25) with $V = D$, $S = \partial D$, which holds for any functions v_1, v_2 defined on a volume V with smooth closed connected surface S. Since $v_1 = 0 = v_2$ on ∂D and $\nabla^2 v_1 = -\lambda_1 v_1$ and $\nabla^2 v_2 = -\lambda_2 v_2$, then Eq. (25) becomes

$$0 = (\lambda_1 - \lambda_2) \int\int\int_D v_1 v_2 dV$$

Thus if $\lambda_1 \neq \lambda_2$,

$$\int\int\int_D v_1 v_2 dV = 0, \tag{32}$$

and the eigenfunctions v_1, v_2 are orthogonal.

7 Heat and Wave problems on a 2D rectangle, homogeneous BCs

Ref: Guenther & Lee §10.2, Myint-U & Debnath §9.4 – 9.6

[Nov 7, 2006]

7.1 Sturm-Liouville Problem on a 2D rectangle

We now consider the special case where the subregion D is a rectangle

$$D = \{(x, y) : 0 \le x \le x_0, \qquad 0 \le y \le y_0\}$$

The Sturm-Liouville Problem (19) becomes

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \lambda v &= 0, \qquad (x, y) \in D, & (33) \\ v(0, y) &= v(x_0, y) = 0, \qquad 0 \le y \le y_0, & (34) \\ v(x, 0) &= v(x, y_0) = 0, \qquad 0 \le x \le x_0. & (35) \end{aligned}$$

Note that in the PDE (33), λ is positive a constant (we showed above that λ had to be both constant and positive). We employ separation of variables again, this time in x and y: substituting $v(x,y) = X(x)Y(y)$ into the PDE (33) and dividing by $X(x)Y(y)$ gives

$$\frac{Y''}{Y} + \lambda = -\frac{X''}{X}$$

Since the l.h.s. depends only on y and the r.h.s. only depends on x, both sides must equal a constant, say μ,

$$\frac{Y''}{Y} + \lambda = -\frac{X''}{X} = \mu \tag{36}$$

The BCs (34) and (35) imply

$$\begin{aligned} X(0)Y(y) = X(x_0)Y(y) = 0, &\qquad 0 \le y \le y_0, \\ X(x)Y(0) = X(x)Y(y_0) = 0, &\qquad 0 \le x \le x_0. \end{aligned}$$

To have a non-trivial solution, $Y(y)$ must be nonzero for some $y \in [0, y_0]$ and $X(x)$ must be nonzero for some $x \in [0, x_0]$, so that to satisfy the previous 2 equations, we must have

$$X(0) = X(x_0) = Y(0) = Y(y_0) = 0 \tag{37}$$

The problem for $X(x)$ is the 1D Sturm-Liouville problem

$$\begin{aligned} X'' + \mu X &= 0, \qquad 0 \le x \le x_0 \\ X(0) &= X(x_0) = 0 \end{aligned} \tag{38}$$

We solved this problem in the chapter on the 1D Heat Equation. We found that for non-trivial solutions, μ had to be positive and the solution is

$$X_m(x) = a_m \sin\left(\frac{m\pi x}{x_0}\right), \qquad \mu_m = \left(\frac{m\pi}{x_0}\right)^2, \qquad m = 1, 2, 3, ... \tag{39}$$

The problem for $Y(y)$ is

$$\begin{aligned} Y'' + \nu Y &= 0, \qquad 0 \le y \le y_0, \\ Y(0) &= Y(y_0) = 0 \end{aligned} \tag{40}$$

where $\nu = \lambda - \mu$. The solutions are the same as those for (38), with ν replacing μ:

$$Y_n(y) = b_n \sin\left(\frac{n\pi y}{y_0}\right), \qquad \nu_n = \left(\frac{n\pi}{y_0}\right)^2, \qquad n = 1, 2, 3, ... \tag{41}$$

The eigen-solution of the 2D Sturm-Liouville problem (33) – (35) is

$$v_{mn}(x,y) = X_m(x)Y_n(y) = c_{mn} \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{n\pi y}{y_0}\right), \qquad m, n = 1, 2, 3, ... \tag{42}$$

with eigenvalue

$$\lambda_{mn} = \mu_m + \nu_n = \pi^2 \left(\frac{m^2}{x_0^2} + \frac{n^2}{y_0^2}\right).$$

7.2 Solution to heat equation on 2D rectangle

The heat problem on the 2D rectangle is the special case of (7),

$$\begin{aligned} u_t &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad (x,y) \in D, \qquad t > 0, \\ u(x,y,t) &= 0, \qquad (x,y) \in \partial D, \\ u(x,y,0) &= f(x,y), \qquad (x,y) \in D, \end{aligned}$$

where D is the rectangle $D = \{(x,y) : 0 \le x \le x_0,\ 0 \le y \le y_0\}$. We reverse the separation of variables (9) and substitute solutions (14) and (42) to the $T(t)$ problem (13) and the Sturm-Liouville problem (33) – (35), respectively, to obtain

$$\begin{aligned} u_{mn}(x,y,t) &= A_{mn} \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{n\pi y}{y_0}\right) e^{-\lambda_{mn} t} \\ &= A_{mn} \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{n\pi y}{y_0}\right) e^{-\pi^2\left(\frac{m^2}{x_0^2}+\frac{n^2}{y_0^2}\right)t} \end{aligned}$$

To satisfy the initial condition, we sum over all m, n to obtain the solution, in general form,

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{n\pi y}{y_0}\right) e^{-\lambda_{mn} t} \tag{43}$$

Setting $t = 0$ and imposing the initial condition $u(x,y,0) = f(x,y)$ gives

$$f(x,y) = u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} v_{mn}(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{n\pi y}{y_0}\right)$$

where $v_{mn}(x,y) = \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{n\pi y}{y_0}\right)$ are the eigenfunctions of the 2D Sturm-Liouville problem on a rectangle, (33) – (35). Multiplying both sides by $v_{\hat{m}\hat{n}}(x,y)$ ($\hat{m}$, $\hat{n} = 1, 2, 3, ...$) and integrating over the rectangle D gives

$$\int\int_D f(x,y)\, v_{\hat{m}\hat{n}}(x,y)\, dA = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \int\int_D v_{mn}(x,y)\, v_{\hat{m}\hat{n}}(x,y)\, dA \tag{44}$$

where $dA = dxdy$. Note that

$$\begin{aligned} \int\int_D v_{mn}(x,y)\, v_{\hat{m}\hat{n}}(x,y)\, dA &= \int_0^{x_0} \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{\hat{m}\pi x}{x_0}\right) dx \\ &\quad \times \int_0^{y_0} \sin\left(\frac{n\pi y}{y_0}\right) \sin\left(\frac{\hat{n}\pi y}{y_0}\right) dy \\ &= \begin{cases} x_0 y_0/4, & m = \hat{m} \text{ and } n = \hat{n} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Thus (44) becomes

$$\int\int_D f(x,y)\, v_{\hat{m}\hat{n}}(x,y)\, dA = \frac{A_{\hat{m}\hat{n}}}{4} x_0 y_0$$

Since $\hat{m}$, $\hat{n}$ are dummy variables, we replace $\hat{m}$ by m and $\hat{n}$ by n, and rearrange to obtain

$$\begin{aligned} A_{mn} &= \frac{4}{x_0 y_0} \int\int_D f(x,y)\, v_{\hat{m}\hat{n}}(x,y)\, dA \\ &= \frac{4}{x_0 y_0} \int_0^{x_0} \int_0^{y_0} f(x,y) \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{n\pi y}{y_0}\right) dy dx \end{aligned} \tag{45}$$

7.3 Solution to wave equation on 2D rectangle, homogeneous BCs

The solution to the wave equation on the 2D rectangle follows similarly. The general 3D wave problem (8) becomes

$$\begin{aligned} u_{tt} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad (x,y) \in D, \qquad t > 0, \\ u(x,y,t) &= 0, \qquad (x,y) \in \partial D, \\ u(x,y,0) &= f(x,y), \qquad (x,y) \in D, \\ u_t(x,y,0) &= g(x,y), \qquad (x,y) \in D, \end{aligned}$$

where D is the rectangle $D = \{(x,y) : 0 \le x \le x_0,\ 0 \le y \le y_0\}$. We reverse the separation of variables (9) and substitute solutions (16) and (42) to the $T(t)$ problem (15) and the Sturm Liouville problem (33) – (35), respectively, to obtain

$$\begin{aligned} u_{mn}(x,y,t) &= \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{n\pi y}{y_0}\right) \left(\alpha_{nm} \cos\left(\sqrt{\lambda_{nm}} t\right) + \beta_{nm} \sin\left(\sqrt{\lambda_{nm}} t\right)\right) \\ &= \sin\left(\frac{m\pi x}{x_0}\right) \sin\left(\frac{n\pi y}{y_0}\right) \\ &\quad \times \left(\alpha_{nm} \cos\left(\pi t \sqrt{\frac{m^2}{x_0^2} + \frac{n^2}{y_0^2}}\right) + \beta_{nm} \sin\left(\pi t \sqrt{\frac{m^2}{x_0^2} + \frac{n^2}{y_0^2}}\right)\right) \end{aligned}$$

To satisfy the initial condition, we sum over all m, n to obtain the solution, in general form,

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(x,y,t) \tag{46}$$

Setting $t = 0$ and imposing the initial conditions

$$u(x,y,0) = f(x,y), \qquad u_t(x,y,0) = g(x,y)$$

gives

$$\begin{aligned} f(x,y) &= u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}v_{mn}(x,y) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}\sin\left(\frac{m\pi x}{x_0}\right)\sin\left(\frac{n\pi y}{y_0}\right) \\ g(x,y) &= u_t(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{\lambda_{mn}}\beta_{mn}v_{mn}(x,y) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{\lambda_{mn}}\beta_{mn}\sin\left(\frac{m\pi x}{x_0}\right)\sin\left(\frac{n\pi y}{y_0}\right) \end{aligned}$$

where $v_{mn}(x,y) = \sin\left(\frac{m\pi x}{x_0}\right)\sin\left(\frac{n\pi y}{y_0}\right)$ are the eigenfunctions of the 2D Sturm Liouville problem on a rectangle, (33) – (35). As above, multiplying both sides by $v_{\hat{m}\hat{n}}(x,y)$ $(\hat{m},\ \hat{n} = 1,2,3,...)$ and integrating over the rectangle D gives

$$\begin{aligned} \alpha_{mn} &= \frac{4}{x_0 y_0}\int\int_D f(x,y)\,v_{mn}(x,y)\,dxdy \\ \beta_{mn} &= \frac{4}{x_0 y_0\sqrt{\lambda_{mn}}}\int\int_D g(x,y)\,v_{mn}(x,y)\,dxdy \end{aligned}$$

8 Heat and Wave equations on a 2D circle, homogeneous BCs

Ref: Guenther & Lee §10.2, Myint-U & Debnath §9.4, 9.13 (exercises)

We now consider the special case where the subregion D is the unit circle (we may assume the circle has radius 1 by choosing the length scale l for the spatial coordinates as the original radius):

$$D = \left\{(x,y) : x^2 + y^2 \leq 1\right\}$$

The Sturm-Liouville Problem (19) becomes

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \lambda v = 0, \qquad (x,y) \in D, \tag{47}$$

$$v(x,y) = 0, \qquad x^2 + y^2 = 1, \tag{48}$$

where we already know λ is positive and real. It is natural to introduce polar coordinates via the transformation

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad w(r,\theta,t) = u(x,y,t)$$

for

$$0 \leq r \leq 1, \qquad -\pi \leq \theta < \pi.$$

You can verify that

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}$$

The PDE becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \lambda w = 0, \qquad 0 \le r \le 1,\ -\pi \le \theta < \pi \tag{49}$$

The BC (48) requires

$$w(1,\theta) = 0, \qquad -\pi \le \theta < \pi. \tag{50}$$

We use separation of variables by substituting

$$w(r,\theta) = R(r)\,H(\theta) \tag{51}$$

into the PDE (49) and multiplying by $r^2/(R(r)\,H(\theta))$ and then rearranging to obtain

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\frac{1}{R(r)} + \lambda r^2 = -\frac{d^2 H}{d\theta^2}\frac{1}{H(\theta)}$$

Again, since the l.h.s. depends only on r and the r.h.s. on θ, both must be equal to a constant μ,

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\frac{1}{R(r)} + \lambda r^2 = -\frac{d^2 H}{d\theta^2}\frac{1}{H(\theta)} = \mu \tag{52}$$

The BC (50) becomes

$$w(1,\theta) = R(1)\,H(\theta) = 0$$

which, in order to obtain non-trivial solutions ($H(\theta) \neq 0$ for some θ), implies

$$R(1) = 0 \tag{53}$$

In the original (x,y) coordinates, it is assumed that $v(x,y)$ is smooth (i.e. continuously differentiable) over the circle. When we change to polar coordinates, we need to introduce an extra condition to guarantee the smoothness of $v(x,y)$, namely, that

$$w(r,-\pi) = w(r,\pi), \qquad w_\theta(r,-\pi) = w_\theta(r,\pi). \tag{54}$$

Substituting (51) gives

$$H(-\pi) = H(\pi), \qquad \frac{dH}{d\theta}(-\pi) = \frac{dH}{d\theta}(\pi). \tag{55}$$

The solution $v(x,y)$ is also bounded on the circle, which implies $R(r)$ must be bounded for $0 \le r \le 1$.

The problem for $H(\theta)$ is

$$\frac{d^2H}{d\theta^2} + \mu H(\theta) = 0; \qquad H(-\pi) = H(\pi), \qquad \frac{dH}{d\theta}(-\pi) = \frac{dH}{d\theta}(\pi). \tag{56}$$

You can show that for $\mu < 0$, we only get the trivial solution $H(\theta) = 0$. For $\mu = 0$, we have $H(\theta) = const$, which works. For $\mu > 0$, non-trivial solutions are found only when $\mu = m^2$,

$$H_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta)$$

Thus, in general, we may assume $\lambda = m^2$, for $m = 0, 1, 2, 3, ...$

The equation for $R(r)$ in (52) becomes

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\frac{1}{R(r)} + \lambda r^2 = \mu = m^2, \qquad m = 0, 1, 2, 3, ...$$

Rearranging gives

$$r^2\frac{d^2R_m}{dr^2} + r\frac{dR_m}{dr} + \left(\lambda r^2 - m^2\right)R_m = 0; \qquad R_m(1) = 0, \qquad |R_m(0)| < \infty \tag{57}$$

We know already that $\lambda > 0$, so we can let

$$s = \sqrt{\lambda}r, \qquad \bar{R}_m(s) = R_m(r)$$

so that (57) becomes

$$s^2\frac{d^2\bar{R}_m}{ds^2} + s\frac{d\bar{R}_m}{ds} + \left(s^2 - m^2\right)\bar{R}_m = 0; \qquad \bar{R}_m\left(\sqrt{\lambda}\right) = 0, \qquad \left|\bar{R}_m(0)\right| < \infty \tag{58}$$

The ODE is called Bessel's Equation which, for each $m = 0, 1, 2, ...$ has two linearly independent solutions, $J_m(s)$ and $Y_m(s)$, called the Bessel functions of the first and second kinds, respectively, of order m. The function $J_m(s)$ is bounded at $s = 0$; the function $Y_m(s)$ is unbounded at $s = 0$. The general solution to the ODE is $\bar{R}_m(s) = c_{m1}J_m(s) + c_{m2}Y_m(s)$ where c_{mn} are constants of integration. Our boundedness criterion $\left|\bar{R}_m(0)\right| < \infty$ at $s = 0$ implies $c_{2m} = 0$. Thus

$$\bar{R}_m(s) = c_m J_m(s), \qquad R_m(r) = c_m J_m\left(\sqrt{\lambda}r\right).$$

[Nov 9, 2006]

The Bessel Function $J_m(s)$ of the first kind of order m has power series

$$J_m(s) = \sum_{k=0} \frac{(-1)^k s^{2k+m}}{k!\,(k+m)!2^{2k+m}}. \tag{59}$$

$J_m(s)$ can be expressed in many ways, see Handbook of Mathematical Functions by Abramowitz and Stegun, for tables, plots, and equations. The fact that $J_m(s)$ is expressed as a power series is not a drawback. It is like $\sin(x)$ and $\cos(x)$, which are also associated with power series. Note that the power series (59) converges absolutely for all $s \geq 0$ and converges uniformly on any closed set $s \in [0, L]$. To see this, note that each term in the sum satisfies

$$\left|\frac{(-1)^k s^{2k+m}}{k!\,(k+m)!2^{2k+m}}\right| \leq \frac{L^{2k+m}}{k!\,(k+m)!2^{2k+m}}$$

Note that the sum of numbers

$$\sum_{k=0} \frac{L^{2k+m}}{k!\,(k+m)!2^{2k+m}}$$

converges by the Ratio Test, since the ratio of successive terms in the sum is

$$\left|\frac{\frac{L^{2(k+1)+m}}{(k+1)!(k+1+m)!2^{2(k+1)+m}}}{\frac{L^{2k+m}}{k!(k+m)!2^{2k+m}}}\right| = \frac{L^2}{(k+1)(k+1+m)4} \leq \frac{L^2}{(k+1)^2 4} = \left(\frac{L}{2(k+1)}\right)^2$$

Thus for $k > N = \lceil L/2 \rceil$,

$$\left|\frac{\frac{L^{2(k+1)+m}}{(k+1)!(k+1+m)!2^{2(k+1)+m}}}{\frac{L^{2k+m}}{k!(k+m)!2^{2k+m}}}\right| < \left(\frac{L}{2(N+1)}\right)^2 < 1$$

Since the upper bound is less than one and is independent of the summation index k, then by the Ratio test, the sum converges absolutely. By the Weirstrass M-Test, the infinite sum in (59) converges uniformly on $[0, L]$. Since L is arbitrary, the infinite sum in (59) converges uniformly on any closed subinterval $[0, L]$ of the real axis.

Each Bessel function $J_m(s)$ has an infinite number of zeros (roots) for $s > 0$. Let $j_{m,n}$ be the n'th zero of the function $J_m(s)$. Note that

	1	2	3
$J_0(s)$	$j_{0,1} = 2.4048$	$j_{0,2} = 5.5001$	$j_{0,3} = 8.6537$
$J_1(s)$	$j_{1,1} = 3.852$	$j_{1,2} = 7.016$	$j_{1,3} = 10.173$

The second BC requires

$$R_m(1) = \bar{R}_m\left(\sqrt{\lambda}\right) = J_m\left(\sqrt{\lambda}\right) = 0$$

This has an infinite number of solutions, namely $\sqrt{\lambda} = j_{m,n}$ for $n = 1, 2, 3, \ldots$. Thus the eigenvalues are

$$\lambda_{mn} = j_{m,n}^2, \quad m, n = 1, 2, 3, \ldots$$

with corresponding eigenfunctions $J_m(rj_{m,n})$. The separable solutions are thus

$$v_{mn}(x,y) = w_{mn}(r,\theta) = \begin{cases} J_0(rj_{0,n}) & n = 1, 2, 3, \ldots \\ J_m(rj_{m,n})(\alpha_{mn}\cos(m\theta) + +\beta_{mn}\sin(m\theta)) & m, n = 1, 2, 3, \ldots \end{cases} \tag{60}$$

8.1 Solution to heat equation on the 2D circle, homogeneous BCs

The heat problem on the 2D circle is the special case of (7),

$$\begin{aligned} u_t &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad (x,y) \in D, \qquad t > 0, \\ u(x,y,t) &= 0, \qquad (x,y) \in \partial D, \\ u(x,y,0) &= f(x,y), \qquad (x,y) \in D, \end{aligned}$$

where D is the circle $D = \{(x,y) : x^2 + y^2 \leq 1\}$. We reverse the separation of variables (9) and substitute solutions (14) and (42) to the $T(t)$ problem (13) and the Sturm Liouville problem (47) – (48), respectively, to obtain

$$u_{mn}(x,y,t) = v_{mn}(x,y)\, e^{-\lambda_{mn} t} = v_{mn}(x,y)\, e^{-J_{m,n}^2 t}$$

where v_{mn} is given in (60).

To satisfy the initial condition, we sum over all m, n to obtain the solution, in general form,

$$\begin{aligned} u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{mn}(x,y)\, e^{-\lambda_{mn} t} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(r j_{m,n})\left(\alpha_{mn}\cos(m\theta) + \beta_{mn}\sin(m\theta)\right) e^{-\lambda_{mn} t} \end{aligned} \tag{61}$$

where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$. Setting $t = 0$ and imposing the initial condition $u(x,y,0) = f(x,y)$ gives

$$f(x,y) = u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{mn}(x,y)$$

We can use orthogonality relations to find α_{mn} and β_{mn}.

9 The Heat Problem on a square with inhomogeneous BC

We now consider the case of the heat problem on the 2D unit square

$$D = \{(x,y) : 0 \leq x, y \leq 1\}, \tag{62}$$

where a hot spot exists on the left side,

$$\begin{aligned} u_t &= \nabla^2 u, \qquad (x,y) \in D \\ u(x,y,t) &= \begin{cases} u_0/\varepsilon, & \{x=0, |y-y_0| < \varepsilon/2\} \\ 0, & \text{otherwise on } \partial D \end{cases} \\ u(x,y,0) &= f(x,y) \end{aligned} \tag{63}$$

where the hot spot is confined to the left side: $0 \le y_0 - \varepsilon/2 \le y \le y_0 + \varepsilon/2 \le 1$.

9.1 Equilibrium solution and Laplace's Eq. on a rectangle

Ref: Guenther & Lee §8.1, 8.2, Myint-U & Debnath §6.6, 8.7

As in the 1D case, we first find the equilibrium solution $u_E(x,y)$, which satisfies the PDE and the BCs,

$$\begin{aligned} \nabla^2 u_E &= 0, \qquad (x,y) \in D \\ u_E(x,y) &= \begin{cases} u_0/\varepsilon & \{x=0, |y-y_0| < \varepsilon/2\} \\ 0 & \text{otherwise on } \partial D \end{cases} \end{aligned}$$

The PDE for u_E is called Laplace's equation. Laplace's equation is an example of an elliptic PDE. The wave equation is an example of a hyperbolic PDE. The heat equation is a parabolic PDE. These are the three types of second order (i.e. involving double derivatives) PDEs: elliptic, hyperbolic and parabolic.

We proceed via separation of variables: $u_E(x,y) = X(x)Y(y)$, so that the PDE becomes

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda$$

where λ is constant since the l.h.s. depends only on x and the middle only on y. The BCs are

$$Y(0) = Y(1) = 0, \qquad X(1) = 0$$

and

$$X(0)Y(y) = \begin{cases} u_0/\varepsilon & |y-y_0| < \varepsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

We first solve for $Y(y)$, since we have 2 easy BCs:

$$Y'' + \lambda Y = 0; \qquad Y(0) = Y(1) = 0$$

The non-trivial solutions, as we have found before, are $Y_n = \sin(n\pi y)$ with $\lambda_n = n^2\pi^2$, for each $n = 1, 2, 3, ...$ Now we consider $X(x)$:

$$X'' - n^2\pi^2 X = 0$$

and hence

$$X(x) = c_1 e^{n\pi x} + c_2 e^{-n\pi x}$$

An equivalent and more convenient way to write this is

$$X(x) = c_3 \sinh n\pi (1-x) + c_4 \cosh n\pi (1-x)$$

Imposing the BC at $x = 1$ gives

$$X(1) = c_4 = 0$$

and hence

$$X(x) = c_3 \sinh n\pi (1-x)$$

Thus the equilibrium solution to this point is

$$u_E(x, y) = \sum_{n=1}^{\infty} A_n \sinh (n\pi (1-x)) \sin (n\pi y)$$

You can check that this satisfies the BCs on $x = 1$ and $y = 0, 1$. Also, from the BC on $x = 0$, we have

$$u_E(0, y) = \sum_{n=1}^{\infty} A_n \sinh (n\pi) \sin (n\pi y) = \begin{cases} u_0/\varepsilon & |y - y_0| < \varepsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

Multiplying both sides by $\sin(m\pi y)$ an integrating in y gives

$$\sum_{n=1}^{\infty} A_n \sinh (n\pi) \int_0^1 \sin (n\pi y) \sin (m\pi y)\, dy = \int_{y_0-\varepsilon/2}^{y_0+\varepsilon/2} \frac{u_0}{\varepsilon} \sin (m\pi y)\, dy$$

From the orthogonality of sin's, we have

$$A_m \sinh (m\pi) \frac{1}{2} = \int_{y_0-\varepsilon/2}^{y_0+\varepsilon/2} \frac{u_0}{\varepsilon} \sin (m\pi y)\, dy$$

Thus,

$$\begin{aligned} A_m &= \frac{2u_0}{\varepsilon \sinh (m\pi)} \int_{y_0-\varepsilon/2}^{y_0+\varepsilon/2} \sin (m\pi y)\, dy \\ &= \frac{2u_0}{\varepsilon \sinh (m\pi)} \left[-\frac{\cos (m\pi y)}{m\pi} \right]_{y_0-\varepsilon/2}^{y_0+\varepsilon/2} \\ &= \frac{2u_0}{\varepsilon m\pi \sinh (m\pi)} \left(\cos (m\pi (y_0 - \varepsilon/2)) - \cos (m\pi (y_0 + \varepsilon/2)) \right) \\ &= \frac{4u_0 \sin (m\pi y_0) \sin \left(\frac{m\pi\varepsilon}{2} \right)}{\varepsilon m\pi \sinh (m\pi)} \end{aligned}$$

Thus

$$u_E(x,y)=\frac{4u_0}{\varepsilon\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi y_0)\sin\left(\frac{n\pi\varepsilon}{2}\right)}{n\sinh(n\pi)}\sinh(n\pi(1-x))\sin(n\pi y)$$

To solve the transient problem, we proceed as in 1-D by defining the function

$$v(x,y,t)=u(x,y,t)-u_E(x,y)$$

so that $v(x,y,t)$ satisfies

$$\begin{aligned} v_t &= \nabla^2 v \\ v &= 0 \qquad \text{on} \qquad \partial D \\ v(x,y,0) &= f(x,y)-u_E(x,y) \end{aligned}$$

9.2 First term approximation

To approximate the equilibrium solution $u_E(x,y)$, note that

$$\frac{\sinh n\pi(1-x)}{\sinh n\pi}=\frac{e^{n\pi(1-x)}-e^{-n\pi(1-x)}}{e^{n\pi}-e^{-n\pi}}$$

For sufficiently large n, we have

$$\frac{\sinh n\pi(1-x)}{\sinh n\pi}\approx\frac{e^{n\pi(1-x)}}{e^{n\pi}}=e^{-n\pi x}$$

Thus the terms decrease in magnitude ($x>0$) and hence $u_E(x,y)$ can be approximated the first term in the series,

$$u_E(x,y)\approx\frac{4u_0}{\varepsilon\pi}\frac{\sin(\pi y_0)\sin\left(\frac{\pi\varepsilon}{2}\right)}{\sinh(\pi)}\sinh(\pi(1-x))\sin(\pi y)$$

A plot of $\sinh(\pi(1-x))\sin(\pi y)$ is given below. The temperature in the center of the square is approximately

$$u_E\left(\frac{1}{2},\frac{1}{2}\right)\approx\frac{4u_0}{\varepsilon\pi}\frac{\sin(\pi y_0)\sin\left(\frac{\pi\varepsilon}{2}\right)}{\sinh(\pi)}\sinh\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$

9.3 Easy way to find steady-state temperature at center

For $y_0=1/2$ and $\varepsilon=1$, we have

$$u_E\left(\frac{1}{2},\frac{1}{2}\right)\approx\frac{4u_0}{\pi}\frac{\sinh\left(\frac{\pi}{2}\right)}{\sinh(\pi)}\approx\frac{u_0}{4}.$$

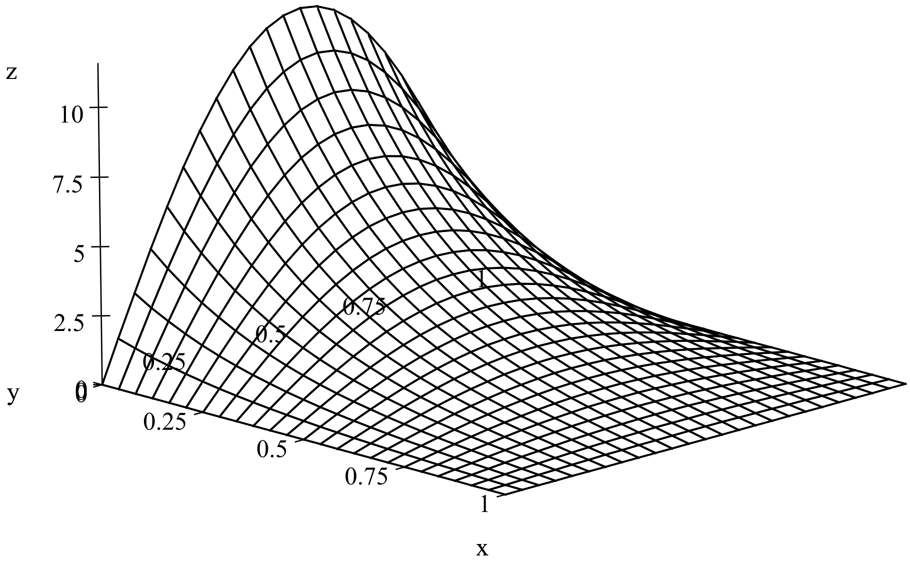

Figure 1: Plot of $\sinh(\pi(1-x))\sin(\pi y)$.

It turns out there is a much easier way to derive this last result. Consider a plate with BCs $u = u_0$ on one side, and $u = 0$ on the other 3 sides. Let $\alpha = u_E\left(\frac{1}{2}, \frac{1}{2}\right)$. Rotating the plate by 90^o will not alter $u_E\left(\frac{1}{2}, \frac{1}{2}\right)$, since this is the center of the plate. Let u_{Esum} be the sums of the solutions corresponding to the BC $u = u_0$ on each of the four different sides. Then by linearity, $u_{Esum} = u_0$ on all sides and hence $u_{Esum} = u_0$ across the plate. Thus

$$u_0 = u_{Esum}\left(\frac{1}{2}, \frac{1}{2}\right) = 4\alpha$$

Hence $\alpha = u_E\left(\frac{1}{2}, \frac{1}{2}\right) = u_0/4$.

9.4 Placement of hot spot for hottest steady-state center

Note that

$$\begin{aligned}
u_E\left(\frac{1}{2}, \frac{1}{2}\right) &\approx \frac{4u_0}{\varepsilon\pi} \frac{\sin(\pi y_0)\sin\left(\frac{\pi\varepsilon}{2}\right)}{\sinh(\pi)} \sinh\left(\frac{\pi}{2}\right) \\
&= \frac{4u_0}{\pi} \frac{\sinh\left(\frac{\pi}{2}\right)}{\sinh(\pi)} \frac{\sin(\pi y_0)\sin\left(\frac{\pi\varepsilon}{2}\right)}{\varepsilon} \\
&\approx \frac{u_0}{4}\sin(\pi y_0)\frac{2}{\pi}\left[\frac{\sin\left(\frac{\pi\varepsilon}{2}\right)}{\frac{\pi\varepsilon}{2}}\right] \\
&\approx \frac{u_0}{4}\sin(\pi y_0)\frac{2}{\pi} \\
&= \frac{u_0}{2\pi}\sin(\pi y_0)
\end{aligned}$$

for small ε. Thus the steady-state center temperature is hottest when the hot spot is placed in the center of the side, i.e. $y_0 = 1/2$.

9.5 Solution to inhomogeneous heat problem on square

We now use the standard trick and solve the inhomogeneous Heat Problem (63),

$$\begin{aligned} u_t &= \nabla^2 u, \qquad (x,y) \in D \\ u(x,y,t) &= \begin{cases} u_0/\varepsilon, & \{x=0, |y-y_0| < \varepsilon/2\} \\ 0, & \text{otherwise on } \partial D \end{cases} \\ u(x,y,0) &= f(x,y) \end{aligned} \tag{64}$$

using the equilibrium solution u_E. We define the transient part of the solution as

$$v(x,y,t) = u(x,y,t) - u_E(x,y)$$

where $u(x,y,t)$ is the solution to (64). The problem for $v(x,y,t)$ is therefore

$$\begin{aligned} v_t &= \nabla^2 v, \qquad (x,y) \in D \\ v(x,y,t) &= 0, \qquad (x,y) \in \partial D \\ v(x,y,0) &= f(x,y) - u_E(x,y) \end{aligned}$$

This is the Heat Problem with homogeneous PDE and BCs. We found the solution to this problem above in (43), (provided we use $x_0 = 1 = y_0$ in (43); don't mix up the side length y_0 in (43) with the location y_0 of the hot spot in the homogeneous problem):

$$v(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi x) \sin(n\pi y) e^{-\pi^2(m^2+n^2)t}$$

where

$$A_{mn} = 4 \int_0^1 \int_0^1 (f(x,y) - u_E(x,y)) \sin(m\pi x) \sin(n\pi y)\, dy dx$$

Thus we have found the full solution $u(x,y,t)$.

10 Heat problem on a circle with inhomogeneous BC

Consider the heat problem

$$\begin{aligned} u_t &= \nabla^2 u, \qquad (x,y) \in D \\ u(x,y,t) &= g(x,y), \qquad (x,y) \in \partial D \\ u(x,y,0) &= f(x,y) \end{aligned} \tag{65}$$

where $D = \{(x,y) : x^2 + y^2 \leq 1\}$ is a circle (disc) of radius 1.

10.1 Equilibrium solution and Laplace's Eq. on a circle

Ref: Guenther & Lee §8.1, 8.2, Myint-U & Debnath §8.4, 8.5

To solve the problem, we must first introduce the steady-state $u = u_E(x,y)$ which satisfies the PDE and BCs,

$$\begin{aligned} \nabla^2 u_E &= 0, & (x,y) \in D, \\ u_E(x,y) &= g(x,y), & (x,y) \in \partial D. \end{aligned}$$

As before, switch to polar coordinates via

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad w_E(r,\theta) = u_E(x,y)$$

for

$$0 \leq r \leq 1, \qquad -\pi \leq \theta < \pi.$$

The problem for u_E becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_E}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w_E}{\partial \theta^2} = 0, \qquad 0 \leq r \leq 1,\ -\pi \leq \theta < \pi \tag{66}$$

$$w(r,-\pi) = w(r,\pi), \qquad w_\theta(r,-\pi) = w_\theta(r,\pi), \tag{67}$$

$$|w(0,\theta)| < \infty \tag{68}$$

$$w(1,\theta) = \hat{g}(\theta), \qquad -\pi \leq \theta < \pi \tag{69}$$

where $\hat{g}(\theta) = g(x,y)$ for $(x,y) \in \partial D$ and $\theta = \arctan(y/x)$.

We separate variables

$$w(r,\theta) = R(r)\,H(\theta)$$

and the PDE becomes

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{d^2H}{d\theta^2}\frac{1}{H(\theta)}$$

Since the l.h.s. depends only on r and the r.h.s. on θ, both must be equal to a constant μ,

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{d^2H}{d\theta^2}\frac{1}{H(\theta)} = \mu$$

The problem for $H(\theta)$ is, as before,

$$\frac{d^2H}{d\theta^2} + \mu H(\theta) = 0; \qquad H(-\pi) = H(\pi), \qquad \frac{dH}{d\theta}(-\pi) = \frac{dH}{d\theta}(\pi),$$

with eigen-solutions

$$H_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta), \qquad m = 0, 1, 2, ...$$

The problem for $R(r)$ is

$$0 = r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - m^2 R = r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - m^2 R$$

Try $R(r) = r^\alpha$ to obtain the auxiliary equation

$$\alpha(\alpha - 1) + \alpha - m^2 = 0$$

whose solutions are $\alpha = \pm m$. Thus for each m, the solution is $R_m(r) = c_1 r^m + c_2 r^{-m}$. For $m > 0$, r^{-m} blows up as $r \to 0$. Our boundedness criterion (68) implies $c_2 = 0$. Hence

$$R_m(r) = c_m r^m$$

where c_m are constants to be found by imposing the BC (69). The separable solutions satisfying the PDE (66) and conditions (67) to (68), are

$$w_m(r,\theta) = r^m \left(A_m \cos(m\theta) + B_m \sin(m\theta)\right), \qquad m = 0, 1, 2, ...$$

The full solution is the infinite sum of these over m,

$$u_E(x,y) = w_E(r,\theta) = \sum_{m=0}^{\infty} r^m \left(A_m \cos(m\theta) + B_m \sin(m\theta)\right) \tag{70}$$

We still need to find the A_m, B_m.

Imposing the BC (69) gives

$$\sum_{m=0}^{\infty} \left(A_m \cos(m\theta) + B_m \sin(m\theta)\right) = \hat{g}(\theta) \tag{71}$$

The orthogonality relations are

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(m\theta)\sin(n\theta)\, d\theta &= 0 \\
\int_{-\pi}^{\pi} \left\{ \begin{array}{c} \cos(m\theta)\cos(n\theta) \\ \sin(m\theta)\sin(n\theta) \end{array} \right\} d\theta &= \left\{ \begin{array}{ll} \pi, & m = n \\ 0, & m \neq n \end{array} \right., \qquad (m > 0).
\end{aligned}$$

Multiplying (71) by $\sin n\theta$ or $\cos n\theta$ and applying these orthogonality relations gives

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{g}(\theta)\, d\theta \\
A_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} \hat{g}(\theta)\cos(m\theta)\, d\theta \\
B_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} \hat{g}(\theta)\sin(m\theta)\, d\theta
\end{aligned} \tag{72}$$

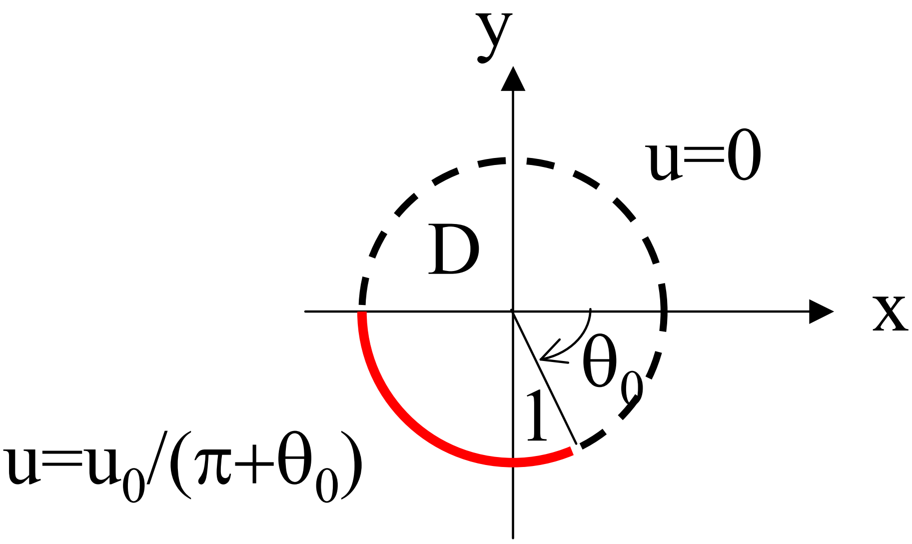

Figure 2: Setup for hot spot problem on circle.

10.1.1 Hot spot on boundary

Suppose

$$\hat{g}(\theta) = \begin{cases} \frac{u_0}{\theta_0+\pi} & -\pi \leq \theta \leq \theta_0 \\ 0 & \text{otherwise} \end{cases}$$

which models a hot spot on the boundary. The Fourier coefficients in (72) are thus

$$\begin{aligned} A_0 &= \frac{u_0}{2\pi}, \qquad A_m = \frac{u_0}{m\pi(\theta_0+\pi)}\sin(m\theta_0) \\ B_m &= -\frac{u_0}{m\pi(\theta_0+\pi)}\left(\cos(m\theta_0) - (-1)^m\right) \end{aligned}$$

Thus the steady-state solution is

$$u_E = \frac{u_0}{2\pi} + \sum_{m=1}^{\infty} r^m \left(\frac{u_0 \sin(m\theta_0)}{m\pi(\theta_0+\pi)}\cos(m\theta) - \frac{u_0\left(\cos(m\theta_0) - (-1)^m\right)}{m\pi(\theta_0+\pi)}\sin(m\theta)\right)$$

10.1.2 Interpretation

The convergence of the infinite series is rapid if $r \ll 1$. If $r \approx 1$, many terms are required for accuracy.

The center temperature ($r = 0$) at equilibrium (steady-state) is

$$u_E(0,0) = w_E(0,\theta) = \frac{u_0}{2\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{g}(\theta)\, d\theta$$

i.e., the mean temperature of the circumference. This is a special case of the Mean Value Property of solutions to Laplace's Equation $\nabla^2 u = 0$.

We now consider plots of $u_E(x, y)$ for some interesting cases. We draw the level curves (isotherms) $u_E = const$ as solid lines. Recall from vector calculus that the gradient of u_E, denoted by ∇u_E, is perpendicular to the level curves. Recall also from the physics that the flux of heat is proportional to ∇u_E. Thus heat flows along the lines parallel to ∇u_E. Note that the heat flows even though the temperature is in steady-state. It is just that the temperature itself at any given point does not change. We call these lines the "heat flow lines" or the "orthogonal trajectories", and draw these as dashed lines in the figure below.

Note that lines of symmetry correspond to (heat) flow lines. To see this, let $\mathbf{n}_l$ be the normal to a line of symmetry. Then the flux at a point on the line is $\nabla u \cdot \mathbf{n}_l$. Rotate the image about the line of symmetry. The arrow for the normal to the line of symmetry is now pointing in the opposite direction, i.e. $-\mathbf{n}_l$, and the flux is $-\nabla u \cdot \mathbf{n}_l$. But since the solution is the same, the flux across the line must still be $\nabla u \cdot \mathbf{n}_l$. Thus

$$\nabla u \cdot \mathbf{n}_l = -\nabla u \cdot \mathbf{n}_l$$

which implies $\nabla u \cdot \mathbf{n}_l = 0$. Thus there is no flux across lines of symmetry. Equivalently, ∇u is perpendicular to the normal to the lines of symmetry, and hence ∇u is parallel to the lines of symmetry. Thus the lines of symmetry are flow lines. Identifying the lines of symmetry help draw the level curves, which are perpendicular to the flow lines. Also, lines of symmetry can be thought of as an insulating boundary, since $\nabla u \cdot \mathbf{n}_l = 0$.

(i) $\theta_0 = 0$. Then

$$u_E = \frac{u_0}{2\pi} - \frac{2u_0}{\pi^2} \sum_{m=1}^{\infty} r^{2n-1} \frac{\sin\left((2n-1)\,\theta\right)}{2n-1}$$

Use the BCs for the boundary. Note that the solution is symmetric with respect to the y-axis (i.e. even in x). The solution is discontinuous at $\{y = 0, x = \pm 1\}$, or $\{r = 1, \theta = 0, \pi\}$. See plot.

(ii) $-\pi < \theta_0 < -\pi/2$. The sum for u_E is messy, so we use intuition. We start with the boundary conditions and use continuity in the interior of the plate to obtain a qualitative idea of the level curves and heat flow lines. See plot.

(iii) $\theta_0 \to -\pi^+$ (a heat spot). Again, use intuition to obtain a qualitative sketch of the level curves and heat flow lines. Note that the temperature at the hot point is infinite. See plot. The heat flux lines must go from a point of hot temperature to a point of low temperature. Since the temperature along boundary is zero except at

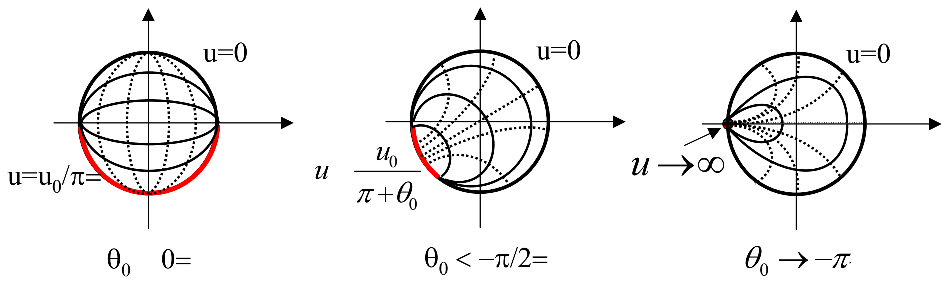

Figure 3: Plots of steady-state temperature due to a hot segment on boundary.

$\theta = -\pi$, then all the heat flux lines must leave the hot spot and extend to various points on the boundary.

10.2 Solution to inhomogeneous heat problem on circle

We now use the standard trick and solve the inhomogeneous Heat Problem (65),

$$\begin{aligned} u_t &= \nabla^2 u, \qquad (x,y) \in D \\ u(x,y,t) &= g(x,y), \qquad (x,y) \in \partial D \\ u(x,y,0) &= f(x,y) \end{aligned} \tag{73}$$

using the equilibrium solution u_E. We define the transient part of the solution as

$$v(x,y,t) = u(x,y,t) - u_E(x,y)$$

where $u(x,y,t)$ is the solution to (73). The problem for $v(x,y,t)$ is therefore

$$\begin{aligned} v_t &= \nabla^2 v, \qquad (x,y) \in D \\ v(x,y,t) &= 0, \qquad (x,y) \in \partial D \\ v(x,y,0) &= f(x,y) - u_E(x,y) \end{aligned}$$

This is the Heat Problem with homogeneous PDE and BCs. We found the solution to this problem above in (61),

$$v(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_m(r j_{m,n}) \left(\alpha_{mn} \cos(m\theta) + \beta_{mn} \sin(m\theta)\right) e^{-\lambda_{mn} t}$$

where α_{mn} and β_{mn} are found from orthogonality relations and $f(x,y) - u_E(x,y)$. Thus we have found the full solution $u(x,y,t)$.

11 Mean Value Property for Laplace's Eq.

Ref: Guenther & Lee §8.4, Myint-U & Debnath §8.4

Theorem [Mean Value Property] Suppose $v(x,y)$ satisfies Laplace's equation in a 2D domain D,

$$\nabla^2 v = 0, \qquad (x,y) \in D. \tag{74}$$

Then at any point (x_0, y_0) in D, v equals the mean value of the temperature around any circle centered at (x_0, y_0) and contained in D,

$$v(x_0, y_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(x_0 + R\cos\theta, y_0 + R\sin\theta)\, d\theta. \tag{75}$$

Note that the curve $\{(x_0 + R\cos\theta, y_0 + R\sin\theta) : -\pi \le \theta < \pi\}$ traces the circle of radius R centered at (x_0, y_0).

Proof: To prove the Mean Value Property, we first consider Laplace's equation (74) on the unit circle centered at the origin $(x,y) = (0,0)$. We already solved this problem, above, when we solved for the steady-state temperature u_E that took the value $\hat{g}(\theta)$ on the boundary. The solution is Eqs. (70) and (72). Setting $r = 0$ in (70) gives the center value

$$u_E(0,0) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{g}(\theta)\, d\theta \tag{76}$$

On the boundary of the circle (of radius 1), $(x,y) = (\cos\theta, \sin\theta)$ and the BC implies that $u_E = \hat{g}(\theta)$ on that boundary. Thus, $\hat{g}(\theta) = u_E(\cos\theta, \sin\theta)$ and (76) becomes

$$u_E(0,0) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_E(\cos\theta, \sin\theta)\, d\theta. \tag{77}$$

We now consider the region

$$B_{(x_0,y_0)}(R) = \left\{(x,y) : (x - x_0)^2 + (y - y_0)^2 \le R^2\right\},$$

which is a disc of radius R centered at $(x,y) = (x_0, y_0)$. Since $B_{(x_0,y_0)}(R) \subseteq D$, Laplace's equation (74) holds on this disc. Thus

$$\nabla^2 v = 0, \qquad (x,y) \in B_{(x_0,y_0)}(R). \tag{78}$$

We make the change of variable

$$\hat{x} = \frac{x - x_0}{R}, \qquad \hat{y} = \frac{y - y_0}{R}, \qquad u_E(\hat{x}, \hat{y}) = v(x,y) \tag{79}$$

to map the circle $B_{(x_0,y_0)}(R)$ into the unit disc $\{(\hat{x}, \hat{y}) : \hat{x}^2 + \hat{y}^2 \le 1\}$. Laplace's equation (78) becomes

$$\hat{\nabla}^2 u_E = 0, \qquad (\hat{x}, \hat{y}) \in \left\{(\hat{x}, \hat{y}) : \hat{x}^2 + \hat{y}^2 \le 1\right\}$$

where $\hat{\nabla}^2 = (\partial^2/\partial\hat{x}^2, \partial^2/\partial\hat{y}^2)$. The solution is given by Eqs. (70) and (72), and we found the center value above in Eq. (77). Reversing the change of variable (79) in Eq. (77) gives

$$v(x_0, y_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(x_0 + R\cos\theta, y_0 + R\sin\theta)\, d\theta$$

as required. ■

For the heat equation, the Mean Value Property implies the equilibrium temperature at any point (x_0, y_0) in D equals the mean value of the temperature around any disc centered at (x_0, y_0) and contained in D.

12 Maximum Principle for Laplace's Eq.

Ref: Guenther & Lee §8.4, Myint-U & Debnath §8.2

Theorem [Maximum Principle] Suppose $v(x, y)$ satisfies Laplace's equation in a 2D domain D,

$$\nabla^2 v = 0, \qquad (x, y) \in D.$$

Then the function v takes its maximum and minimum on the boundary of D, ∂D.

Proof: Let (x_0, y_0) be an interior point in D where v takes it's maximum. In particular, (x, y) is not on the boundary ∂D of D. The Mean Value Property implies that for any disc $B_{(x_0,y_0)}(R)$ of radius $R > 0$ centered at (x_0, y_0),

$$\begin{aligned} v(x_0, y_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} v(x_0 + R\cos\theta, y_0 + R\sin\theta)\, d\theta \\ &= \text{Average of } v \text{ on boundary of circle} \end{aligned}$$

But the average is always between the minimum and maximum. Thus $v(x_0, y_0)$ must be between the minimum and maximum value of $v(x, y)$ on the boundary of the disc. But since $v(x_0, y_0)$ is the maximum of v on D, then the entire boundary of the circle must have value $v(x_0, y_0)$. Since $R > 0$ is arbitrary, this holds for all values in the disc $B_{(x_0,y_0)}(R)$. Keep increasing R until the disc hits the boundary of D. Then $v(x_0, y_0)$ is a value of v along the boundary of D.

A similar argument holds for the minimum of v. ■

Corollary If $v = 0$ everywhere on the boundary, then v must be zero at everywhere in D.

For the homogeneous heat equation (i.e. no sinks/sources), this implies the equilibrium temperature attains its maximum/minimum on the boundary.

If the boundary is completely insulated, $\nabla v = 0$ on ∂D, then the equilibrium temperature is constant. To see this, we apply result (26) to the case where v is the solution to Laplace's equation with $\nabla v = 0$ on ∂D,

$$0 = \int\int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS = \int\int\int_D \left(v\nabla^2 v + |\nabla v|^2\right) dV = \int\int\int_D |\nabla v|^2 \, dV$$

Thus, by continuity, $|\nabla v| = 0$ and $\nabla v = 0$ everywhere in D and hence $v = \mathit{const}$ in D.

13 Eigenvalues on different domains

In this section we revisit the Sturm-Liouville problem

$$\begin{aligned} \nabla^2 v + \lambda v &= 0, \qquad \mathbf{x} \in D \\ v &= 0, \qquad \mathbf{x} \in \partial D \end{aligned}$$

and consider the effect of the shape of the domain on the eigenvalues.

Definition Rayleigh Quotient

$$R(v) = \frac{\int\int\int_D \nabla v \cdot \nabla v dV}{\int\int\int_D v^2 dV} \tag{80}$$

Theorem Given a domain $D \subseteq \mathbb{R}^3$ and any function v that is piecewise smooth on D, non-zero at some points on the interior of D, and zero on all of ∂D, then the smallest eigenvalue of the Laplacian on D satisfies

$$\lambda \leq R(v)$$

and $R(h) = \lambda$ if and only if $h(\mathbf{x})$ is an eigen-solution of the Sturm Liouville problem on D.

Sketch Proof: We use result (26) derived for any smooth function v defined on a volume V with closed smooth surface S.

$$\int\int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS = \int\int\int_D \left(v\nabla^2 v + \nabla v \cdot \nabla v\right) dV$$

In the statement of the theorem, we assumed that $v = 0$ on ∂D, and hence

$$\int\int\int_D \nabla v \cdot \nabla v dV = -\int\int\int_D v\nabla^2 v dV \tag{81}$$

Let $\{\phi_n\}$ be an orthonormal basis of eigenfunctions on D, i.e. all the functions ϕ_n which satisfy

$$\begin{aligned} \nabla^2 \phi_n + \lambda_n \phi_n &= 0, \qquad \mathbf{x} \in D \\ \phi_n &= 0, \qquad \mathbf{x} \in \partial D \end{aligned}$$

and

$$\int\int\int_D \phi_n \phi_m dV = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

We can expand v in the eigenfunctions,

$$v(\mathbf{x}) = \sum_{n=1}^{\infty} A_n \phi_n(\mathbf{x})$$

where the A_n are constants. Assuming we can differentiate the sum termwise, we have

$$\nabla^2 v = \sum_{n=1}^{\infty} A_n \nabla^2 \phi_n = -\sum_{n=1}^{\infty} \lambda_n A_n \phi_n \tag{82}$$

The orthonormality property (i.e., the orthogonality property with $\int\int\int_D \phi_n^2 dV = 1$) implies

$$\int\int\int_D v^2 dV = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n A_m \int\int\int_D \phi_n \phi_m dV = \sum_{n=1}^{\infty} A_n^2 \tag{83}$$

and, from (82),

$$\int\int\int_D v \nabla^2 v dV = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n A_n A_m \int\int\int_D \phi_n \phi_m dV = -\sum_{n=1}^{\infty} \lambda_n A_n^2 \tag{84}$$

Substituting (84) into (81) gives

$$\int\int\int_D \nabla v \cdot \nabla v dV = -\int\int\int_D v \nabla^2 v dV = \sum_{n=1}^{\infty} \lambda_n A_n^2 \tag{85}$$

Substituting (83) and (85) into (80) gives

$$R(v) = \frac{\int\int\int_D \nabla v \cdot \nabla v dV}{\int\int\int_D v^2 dV} = \frac{\sum_{n=1}^{\infty} \lambda_n A_n^2}{\sum_{n=1}^{\infty} A_n^2}$$

We assume the eigenfunctions are arranged in increasing order. In particular, $\lambda_n \geq \lambda_1$. Thus

$$R(v) \geq \frac{\sum_{n=1}^{\infty} \lambda_1 A_n^2}{\sum_{n=1}^{\infty} A_n^2} = \lambda_1 \frac{\sum_{n=1}^{\infty} A_n^2}{\sum_{n=1}^{\infty} A_n^2} = \lambda_1.$$

If v is an eigenfunction with eigenvalue λ_1, then

$$R(v) = \frac{\lambda_1 A_1^2}{A_1^2} = \lambda_1$$

If v is not an eigenfunction corresponding to λ_1, then there exists an $n > 1$ such that $\lambda_n > \lambda_1$ and $A_n \neq 0$, so that $R(v) > \lambda_1$. ■

Theorem If two domains $\widehat{D}$ and D in $\mathbb{R}^2$ satisfy

$$D \subsetneq \widehat{D} \qquad \text{i.e., } D \subset \widehat{D} \text{ but } D \neq \widehat{D},$$

then the smallest eigenvalues of the Sturm-Liouville problems on D and $\widehat{D}$, λ_1 and $\hat{\lambda}_1$, respectively, satisfy

$$\widehat{\lambda}_1 < \lambda_1$$

In other words, the domain $\widehat{D}$ that contains the sub-domain D is associated with a smaller eigenvalue.

Proof: Note that the Sturm-Liouville problems are

$$\begin{aligned} \nabla^2 v + \lambda v &= 0, \qquad (x, y) \in D \\ v &= 0, \qquad (x, y) \in \partial D \end{aligned}$$

$$\begin{aligned} \nabla^2 \widehat{v} + \widehat{\lambda} \widehat{v} &= 0, \qquad (x, y) \in \widehat{D} \\ \widehat{v} &= 0, \qquad (x, y) \in \partial \widehat{D} \end{aligned}$$

Let v_1 be the eigenfunction corresponding to λ_1 on D. Then, as we have proven before,

$$\lambda_1 = R\left(v_1\right), \tag{86}$$

where $R(v)$ is the Rayleigh Quotient. Extend the function v_1 continuously from D to $\widehat{D}$ to obtain a function $\hat{v}_1$ on $\widehat{D}$ which satisfies

$$\hat{v}_1 = \begin{cases} v_1\left(x, y\right), & (x, y) \in D \\ 0 & (x, y) \in \widehat{D}, \quad (x, y) \notin D \end{cases}$$

The extension is continuous, since v_1 is zero on the boundary of D. Applying the previous theorem to the region $\widehat{D}$ and function $\hat{v}_1$ (which satisfies all the requirements of the theorem) gives

$$\hat{\lambda}_1 \leq R\left(\hat{v}_1\right).$$

Equality happens only if $\hat{v}_1$ is the eigenfunction corresponding to $\hat{\lambda}_1$.

Useful fact [stated without proof]: the eigenfunction(s) corresponding to the smallest eigenvalue $\hat{\lambda}_1$ on $\widehat{D}$ are nonzero in the interior of $\widehat{D}$.

From the useful fact, $\hat{v}_1$ cannot be an eigenfunction corresponding to $\hat{\lambda}_1$ on $\widehat{D}$, since it is zero in the interior of $\widehat{D}$ (outside D). Thus, as the previous theorem states,

$$\hat{\lambda}_1 < R\left(\hat{v}_1\right). \tag{87}$$

Since $\hat{v}_1 = 0$ outside D, the integrals over $\widehat{D}$ in the Rayleigh quotient reduce to integrals over D, where $\hat{v}_1 = v_1$, and hence

$$R(\hat{v}_1) = R(v_1). \tag{88}$$

Combining (86), (87), and (88) gives the result,

$$\hat{\lambda}_1 < \lambda_1.$$

■

Example: Consider two regions, D_1 is a rectangle of length x_0, height y_0 and D_2 is a circle of radius R. Recall that the smallest eigenvalue on the rectangle D_1 is

$$\lambda_{11} = \pi^2 \left(\frac{1}{x_0^2} + \frac{1}{y_0^2} \right)$$

The smallest eigenvalue on the circle of radius 1 is $\lambda_{01} = J_{0,1}^2$ where the first zero of the Bessel function $J_0(s)$ of the first kind is $J_{0,1} = 2.4048$. Since $\sqrt{\lambda}$ multiplied r in the Bessel function, then for a circle of radius R, we'd rescale by the change of variable $\hat{r} = r/R$, so that $J_m\left(\sqrt{\lambda} r\right) = J_m\left(\sqrt{\frac{\lambda}{R^2}}\hat{r}\right)$ where $\hat{r}$ goes from 0 to 1. Thus on the circle of radius R, the smallest eigenvalue is

$$\lambda_{01} = \left(\frac{J_{0,1}}{R} \right)^2, \qquad J_{0,1} = 2.4048.$$

Suppose the rectangle is actually a square of side length $2R$. Then

$$\lambda_{11} = \frac{\pi^2}{2R^2} = \frac{4.934}{R^2}, \qquad \lambda_{01} = \left(\frac{J_{0,1}}{R} \right)^2 = \frac{5.7831}{R^2}$$

Thus, $\lambda_{11} < \lambda_{01}$, which confirms the second theorem, since $D_2 \subset D_1$, i.e., the circle is contained inside the square.

Now consider the function

$$v(r) = 1 - \left(\frac{r}{R} \right)^2$$

You can show that

$$\nabla v \cdot \nabla v = \left(\frac{dv}{dr} \right)^2,$$

and

$$R(v) = \frac{\int\int_{D_2} \nabla v \cdot \nabla v dA}{\int\int_{D_2} v^2 dA} = \frac{\int_{-\pi}^{\pi} \int_0^R \left(\frac{dv}{dr}\right)^2 r dr d\theta}{\int_{-\pi}^{\pi} \int_0^R v^2 r dr d\theta} = \frac{6}{R^2} > \lambda_{01}$$

This confirms the first theorem, since $v(r)$ is smooth on D_2, $v(R) = 0$ (zero on the boundary of D_2), and v is nonzero in the interior.

13.1 Faber-Kahn inequality

Thinking about the heat problem on a 2D plate, what shape of plate will cool the slowest? It is a geometrical fact that of all shapes of equal area, the circle (disc) has the smallest circumference. Thus, on physical grounds, we expect the circle to cool the slowest. Faber and Kahn proved this in the 1920s.

Faber-Kahn inequality For all domains $D \subset \mathbb{R}^2$ of equal area, the disc has the smallest first eigenvalue λ_1.

Example. Consider the circle of radius 1 and the square of side length $\sqrt{\pi}$. Then both the square and circle have the same area. The first eigenvalues for the square and circle are, respectively,

$$\lambda_{1SQ} = \pi^2 \left(\frac{1}{\pi} + \frac{1}{\pi} \right) = 2\pi = 6.28, \qquad \lambda_{1CIRC} = (J_{0,1})^2 = 5.7831$$

and hence $\lambda_{1SQ} > \lambda_{1CIRC}$, as the Faber-Kahn inequality states.

14 Nodal lines

Consider the Sturm-Liouville problem

$$\begin{aligned} \nabla^2 v + \lambda v &= 0, \qquad \mathbf{x} \in D \\ v &= 0, \qquad \mathbf{x} \in \partial D \end{aligned} \tag{89}$$

Nodal lines are the curves where the eigenfunctions of the Sturm-Liouville problem are zero. For the solution to the vibrating membrane problem, the normal modes $u_{nm}(x, y, t)$ are zero on the nodal lines, for all time. These are like nodes on the 1D string. Here we consider the nodal lines for the square and the disc (circle).

14.1 Nodal lines for the square

For the square, the eigenfunctions and eigenvalues are a special case of those we found for the rectangle, with side length $x_0 = y_0 = a$,

$$v_{mn}(x, y) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right), \qquad \lambda_{mn} = \frac{\pi^2}{a^2}\left(m^2 + n^2\right), \qquad n, m = 1, 2, 3, ...$$

The nodal lines are the lines on which $v_{mn}(x, y) = 0$, and are

$$x = \frac{ka}{m}, \qquad y = \frac{la}{n}, \qquad 1 \le k \le m-1, \qquad 1 \le l \le n-1$$

for $m, n \ge 2$. Note that $v_{11}(x, y)$ has no nodal lines on the interior - it is only zero on boundary ∂D. Since $\lambda_{mn} = \lambda_{nm}$, then the function $f_{nm} = Av_{mn} + Bv_{nm}$ is also an eigenfunction with eigenvalue λ_{mn}, for any constants A, B. The nodal lines for f_{nm} can be quite interesting.

Examples: we draw the nodal lines on the interior and also the lines around the boundary, where $v_{nm} = 0$.

(i) $m = 1$, $n = 1$.

$$v_{11} = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$$

This is positive on the interior and zero on the boundary. Thus the nodal lines are simply the square boundary ∂D.

(ii) $m = 1$, $n = 2$.

$$v_{12} = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)$$

The nodal lines are the boundary ∂D and the horizontal line $y = a/2$.

(iii) $m = 1$, $n = 3$

$$v_{13} = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi y}{a}\right)$$

The nodal lines are the boundary ∂D and the horizontal lines $y = a/3,\ 2a/3$.

(iv) $m = 3$, $n = 1$

$$v_{31} = \sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$$

The nodal lines are the boundary ∂D and the vertical lines $x = a/3,\ 2a/3$.

(v) Consider $v_{13} - v_{31}$. Since $\lambda_{31} = \lambda_{13} = 10\pi^2/a^2$, this is a solution to SL problem (89) on D with $\lambda = \lambda_{13}$. To find the nodal lines, we use the identity $\sin 3\theta = (\sin\theta)\left(3 - 4\sin^2\theta\right)$ to write

$$v_{13} = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\left(3 - 4\sin^2\left(\frac{\pi y}{a}\right)\right)$$

$$v_{31} = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\left(3 - 4\sin^2\left(\frac{\pi x}{a}\right)\right)$$

$$v_{13} - v_{31} = 4\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\left(\sin^2\left(\frac{\pi x}{a}\right) - \sin^2\left(\frac{\pi y}{a}\right)\right)$$

The nodal lines are the boundary of the square, ∂D, and lines such that

$$0 = \sin^2\left(\frac{\pi x}{a}\right) - \sin^2\left(\frac{\pi y}{a}\right) = \left(\sin\left(\frac{\pi x}{a}\right) - \sin\left(\frac{\pi y}{a}\right)\right)\left(\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{\pi y}{a}\right)\right)$$

i.e.,

$$\sin\left(\frac{\pi x}{a}\right) = \sin\left(\pm\frac{\pi y}{a}\right)$$

Thus

$$\frac{\pi x}{a} = \pm\frac{\pi y}{a} + k\pi, \qquad \text{any integer } k.$$

Hence the nodal lines are

$$y = \pm x + ka$$

for all integers k. We're only concerned with the nodal lines that intersect the interior of the square plate:

$$y = x, \qquad y = -x + a$$

Thus the nodal lines of $v_{13} - v_{31}$ are the sides and diagonals of the square.

Let D_T be the isosceles right triangle whose hypotenuse lies on the bottom horizontal side of the square. The function $v_{13} - v_{31}$ is zero on the boundary ∂D_T, positive on the interior of D_T, and is thus the eigenfunction corresponding to the first (smallest) eigenvalue $\lambda = \lambda_{13}$ of the Sturm-Liouville problem (89) on the triangle D_T. In this case, we found the eigenfunction without using separation of variables, which would have been complicated on the triangle.

(vi) With v_{13}, v_{31} given above, adding gives

$$v_{13} + v_{31} = 4 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \left\{\frac{3}{2} - \left(\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{\pi y}{a}\right)\right)\right\}$$

The nodal lines for $v_{13} + v_{31}$ are thus the square boundary and the closed nodal line defined by

$$\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{\pi y}{a}\right) = \frac{3}{2}.$$

Let D_c be the region contained within this closed nodal line. The function $-(v_{13} + v_{31})$ is zero on the boundary ∂D_c, positive on the interior of D_c, and thus $-(v_{13} + v_{31})$ is the eigenfunction corresponding to the first (smallest) eigenvalue λ_{13} of the Sturm-Liouville problem (89) on D_c.

(vii) Find the first eigenvalue on the right triangle

$$D_{T2} = \left\{0 \le y \le \sqrt{3}x, \quad 0 \le x \le 1\right\}.$$

Note that separation of variables is ugly, because you'd have to impose the BC

$$X(x)\, Y\left(\sqrt{3}x\right) = 0$$

We proceed by placing the triangle inside a rectangle of horizontal and vertical side lengths 1 and $\sqrt{3}$, respectively. The sides of the rectangle coincide with the perpendicular sides of the triangle. Thus, all eigenfunctions v_{mn} for the rectangle are already zero on two sides of the triangle. However, any particular eigenfunction v_{mn} will not be zero on the triangle's hypotenuse, since all the nodal lines of v_{mn} are horizontal or vertical. Thus we need to add multiple eigenfunctions. However, to satisfy the Sturm-Liouville problem, all the eigenfunctions must be associated with the same eigenvalue. For the rectangle with side lengths $\sqrt{3}$ and 1, the eigenvalues are given by

$$\lambda_{mn} = \pi^2 \left(\frac{m^2}{1} + \frac{n^2}{3} \right) = \frac{\pi^2}{3} \left(3m^2 + n^2 \right)$$

We create a table of $3m^2 + n^2$ and look for repeated values:

n, m	1	2	3	4
1	4	13	**28**	
2	7	16	31	
3	12	21	36	
4	19	**28**	43	
5	**28**	37		

The smallest repeated value of $3m^2 + n^2$ is 28:

$$\lambda_{31} = \lambda_{24} = \lambda_{15} = 28\pi^2/3$$

The linear combination of the corresponding eigenfunctions is itself an eigenfunction of the SL problem (89) on the rectangle D,

$$v = Av_{31} + Bv_{24} + Cv_{15}.$$

We wish v to also be zero along the hypotenuse of D_{T2}, and seek A, B, C which satisfy

$$Av_{31} + Bv_{24} + Cv_{15} = 0$$

on $y = \sqrt{3}x$. If we can find such constants A, B, C then we have found an eigenfunction of the SL problem (89) on the triangle D_{T2}. If this eigenfunction is positive on the interior of D_{T2} then we have also found the smallest eigenvalue (in this case λ_{31}).

If we cannot find such A, B, C, then we seek the next largest repeated value of $3m^2 + n^2$ and continue as above. If we can find (A, B, C) such that $v = 0$ on $y = \sqrt{3}x$, but v is zero on the interior of D_{T2}, then λ_{31} is not the smallest eigenvalue and we must pursue another method of finding the smallest eigenvalue. The Rayleigh quotient can help us identify an upper bound.

14.2 Nodal lines for the disc (circle)

For the disc of radius 1, we found the eigenfunctions and eigenvalues to be

$$v_{mnS} = J_m \left(r j_{m,n} \right) \sin m\theta, \qquad v_{mnC} = J_m \left(r j_{m,n} \right) \cos m\theta$$

with

$$\lambda_{mn} = \pi^2 j_{m,n}^2, \qquad n, m = 1, 2, 3, ...$$

The nodal lines are the lines on which $v_{mnC} = 0$ or $v_{mnS} = 0$.

Examples.

(i) $m = 0$, $n = 1$.

$$v_{01} = J_0 \left(r j_{0,1} \right)$$

The nodal lines are the boundary of the disc (circle of radius 1).

(ii) $m = 0$, $n = 2$.

$$v_{02} = J_0 \left(r j_{0,2} \right)$$

The nodal lines are two concentric circles, one of radius $r = 1$, the other or radius $r = J_{0,1}/J_{0,2} < 1$.

(iii) $m = 1$, $n = 1$ and sine.

$$v_{11S} = J_0 \left(r j_{1,1} \right) \sin \theta$$

The nodal lines are the boundary (circle of radius 1) and the line $\theta = -\pi,\ 0,\ \pi$ (horizontal diameter).

15 Steady-state temperature in a 3D cylinder

Suppose a 3D cylinder of radius a and height L has temperature $u\left(r, \theta, z, t\right)$. We assume the axis of the cylinder is on the z-axis and $\left(r, \theta, z\right)$ are cylindrical coordinates. Initially, the temperature is $u\left(r, \theta, z, 0\right)$. The ends are kept at a temperature of $u = 0$ and sides kept at $u\left(a, \theta, z, t\right) = g\left(\theta, z\right)$. The steady-state temperature $u_E\left(r, \theta, z\right)$ in the 3D cylinder is given by

$$\begin{aligned} \nabla^2 u_E &= 0, && -\pi \le \theta < \pi,\ 0 \le r \le a,\ 0 \le z \le L, && (90) \\ u_E\left(r, \theta, 0\right) &= u_E\left(r, \theta, L\right) = 0, && -\pi \le \theta < \pi,\ 0 \le r \le a, && (91) \\ u_E\left(a, \theta, z\right) &= g\left(\theta, z\right), && -\pi \le \theta < \pi,\ 0 \le z \le L. \end{aligned}$$

In cylindrical coordinates $\left(r, \theta, z\right)$, the Laplacian operator becomes

$$\nabla^2 v = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2}$$

We separate variables as

$$v\left(r, \theta, z\right) = R\left(r\right) H\left(\theta\right) Z\left(z\right)$$

so that (90) becomes

$$\frac{1}{rR(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)+\frac{1}{r^2H(\theta)}\frac{d^2H(\theta)}{d\theta^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=0.$$

Rearranging gives

$$\frac{1}{rR(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)+\frac{1}{r^2H(\theta)}\frac{d^2H(\theta)}{d\theta^2}=-\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=\lambda \tag{92}$$

where λ is constant since the l.h.s. depends only on r, θ while the middle depends only on z.

The function $Z(z)$ satisfies

$$\frac{d^2Z}{dz^2}+\lambda Z=0$$

The BCs at $z=0$, L imply

$$\begin{aligned} 0 &= u(r,\theta,0)=R(r)\,H(\theta)\,Z(0) \\ 0 &= u(r,\theta,L)=R(r)\,H(\theta)\,Z(L) \end{aligned}$$

To obtain non-trivial solutions, we must have

$$Z(0)=0=Z(L).$$

As we've shown many times before, the solution for $Z(z)$ is, up to a multiplicative constant,

$$Z_n(z)=\sin\left(\frac{n\pi z}{L}\right),\qquad \lambda_n=\left(\frac{n\pi}{L}\right)^2,\qquad n=1,2,3,...$$

Multiplying Eq. (92) by r^2 gives

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)-\lambda_n r^2=-\frac{1}{H(\theta)}\frac{d^2H(\theta)}{d\theta^2}=\mu \tag{93}$$

where μ is constant since the l.h.s. depends only on r and the middle only on θ.

We assume $g(\theta,z)$ is smooth and 2π-periodic in θ. Hence

$$H'(-\pi)=H'(\pi),\qquad H'(-\pi)=H'(\pi)$$

We solved this problem above and found that $\mu=m^2$ for $m=0,1,2,...$, $H_0(\theta)=const$ and

$$H_m(\theta)=A_m\cos m\theta+B_m\sin m\theta,\qquad m=1,2,3,...$$

Multiplying (93) by $R(r)$ gives

$$r\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)-\left(\lambda_n r^2+m^2\right)R(r)=0$$

This is the Modified Bessel Equation with linearly independent solutions $I_m\left(r\sqrt{\lambda_n}\right)$ and $K_m\left(r\sqrt{\lambda_n}\right)$ called the modified Bessel functions of order m of the first and second kinds, respectively. Thus

$$R_{mn}(r) = c_{1m} I_m\left(r\sqrt{\lambda_n}\right) + c_{2m} K_m\left(r\sqrt{\lambda_n}\right)$$

Since the I_m's are regular (bounded) at $r = 0$, while the K_m's are singular (blow up), and since $R_{mn}(r)$ must be bounded, we must have $c_{2m} = 0$, or

$$R_{mn}(r) = c_{1m} I_m\left(r\sqrt{\lambda_n}\right) = c_{1m} I_m\left(\frac{n\pi r}{L}\right)$$

Thus the general solution is, by combining constants,

$$u_E(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m\left(\frac{n\pi r}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\left[A_{mn}\cos(m\theta) + B_m \sin(m\theta)\right]$$

In theory, we can now impose the condition $u(a,\theta,z) = g(\theta,z)$ and find A_{mn}, B_{mn} using orthogonality of sin, cos.