Lecture 6
Singular Points of Ordinary Differential Equations

In the last chapter, we applied the method of separation of variables to various PDEs and found a number of transcendental equations. Two of them are

$$\frac{d^2y}{dx^2} - xy = 0 \qquad \text{(Airy eq.)}$$

and

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0. \qquad \text{(Bessel eq.)}$$

These equations cannot be solved in closed forms.

As one handles real problems in the real world, one commonly encounters equations which cannot be solved in closed forms. In such cases it’s helpful if one is able to find approximate solutions for them. In this chapter, we’ll mainly focus on making approximations for equations which cannot be solved in closed form.

A Taylor Series Solutions

We will first consider the Airy equation

$$y'' - xy = 0. \tag{6.1}$$

Consider the initial value problem in which the values $y(0)$ and $y'(0)$ of the solution of this equation are given.

Let’s express the solution in the Maclaurin series

$$y(x) = \sum a_n x^n \tag{6.4}$$

where the summation ostensibly covers all positive and negative n, but with the understanding that

$$a_{-n} = 0,\ n = 1, 2, \cdots.$$

Thus (6.4) is indeed a Maclaurin series.

We set $n = 3m$ in (6.7); we get

$$a_{3m} = \frac{a_{3(m-1)}}{3^2 m(m - \frac{1}{3})}. \tag{6.8}$$

Note that we have taken care to write

$$(3m - 1) = 3(m - 1/3),$$

making the coefficient of m inside the parenthesis unity. As we shall see, this makes it convenient to express a_{3m} in terms of Gamma functions.

Problem for the Reader:

Keep applying (6.8), each time reducing m by unity, until you succeed in expressing a_{3m} in terms of a_0.

Answer

Since there is a factor 3^2 in the denominator of (6.8), we get an additional factor 3^2 in the denominator each time we apply (6.8). Thus, applying (6.8) m times, we get a factor 3^{2m} in the denominator. Similarly, the factor m in the denominator of (6.8) generates, as we apply (6.8) successively, the factors $m(m-1)(m-2)\cdots$, and similarly for the factor $(m - \frac{1}{3})$ in the denominator of (6.8). Thus we get

$$a_{3m} = \frac{a_0}{3^{2m}[m(m-1)\bullet\bullet\bullet 1]\left[(m-\frac{1}{3})(m-\frac{4}{3})\bullet\bullet\bullet(\frac{2}{3})\right]}. \tag{6.9}$$

The expression (6.9) can be written in a more compact form. First of all, we note that

$$m(m-1)\bullet\bullet\bullet 1 = m!.$$

And, by (A.2) in this chapter’s Appendix

$$(m+a)(m+a-1)\cdots(1+a) = \frac{\Gamma(m+a+1)}{\Gamma(1+a)}, \tag{6.10}$$

where $\Gamma(z)$ is the Gamma function defined by

$$\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1} dt. \tag{6.11}$$

Thus we have

$$a_{3m} = \frac{\Gamma(2/3)}{3^{2m} m! \Gamma(m+2/3)} a_0. \tag{6.12}$$

Problem for the Reader:

Express a_{3m+1} in terms of a_{3m-2}.

Answer

Setting $n = 3m+1$ in (6.7), we get

$$a_{3m+1} = \frac{a_{3(m-1)+1}}{3^2\left(m+\frac{1}{3}\right)(m)}. \tag{6.13}$$

Problem for the Reader:

Express a_{3m+1} in terms of a_1.

Answer

We may derive from (6.13) and (6.10) that

$$a_{3m+1} = \frac{a_1 \Gamma(4/3)}{3^{2m} m! \Gamma(\mathrm{m}+4/3)}. \tag{6.14}$$

Thus the general solution of the Airy equation is

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

where

$$y_1(x) = \sum_m \frac{x^{3m}}{3^{2m} m! \Gamma\left(m+\frac{2}{3}\right)}, \tag{6.15}$$

and

$$y_2(x) = \sum_m \frac{x^{3m+1}}{3^{2m} m! \Gamma\left(m+\frac{4}{3}\right)}, \tag{6.16}$$

with

$$C_1 \equiv a_0 \Gamma(2/3) \text{ and } C_2 \equiv a_1 \Gamma(4/3).$$

If x is small, taking a few terms of (6.15) and (6.16) gives a good numerical approximation of the solution of the Airy equation.

But (6.15) and (6.16) are more useful than that. Indeed, the series can be used for any finite value of x, not just for small x. This is because the series is convergent even for large x.

Homeworks due next Monday;

Problems 1,2,3 and 4 of Chapter 6 in the textbook.