# Course 18.312: Algebraic Combinatorics

## Problem Set # 2

## Due Wednesday February 18th, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) Start with n coins heads up. Choose a coin at random (each equally likely) and turn it over. Do this a total of $\ell$ times.

   (10 points) What is the probability that all coins will have heads up?

   (10 points) What is the probability that all coins will have tails up?

   (10 points) If we turn over two coins at a time, what is the probability that all coins will have heads up?

2) The **Perron-Frobenius Theorem** asserts the following: Let $M$ be a $p \times p$ matrix of nonnegative real numbers, such that some power of $M$ has all positive entries. Then there is a unique eigenvalue $\lambda$ of $M$ of maximum absolute value, and $\lambda$ is real and positive. Moreover, the eigenvector $v$ corresponding to $\lambda$ is the unique eigenvector of $M$ (up to scalar multiplication) all of whose entries are positive.

   Now let $G$ be a (finite) graph with vertices $v_1, \ldots, v_p$. Assume that some power of the "probability matrix" $\mathbf{M}(G)$ defined in Section 3 has positive entries. (It's not hard to see that this is equivalent to $G$ being connected and containing at least one cycle of odd length, but you do not need to show this.) Let $d_k$ denote the degree (number of incident edges) of vertex $v_k$. Let $D = d_1 + d_2 + \ldots + d_p = 2q - r$, where $G$ has $q$ edges and $r$ loops. Start at any vertex of $G$ and do a random walk on the vertices of $G$ as defined in Section 3. Let $p_k(\ell)$ denote the probability of ending up at vertex $v_k$ after $\ell$ steps.

(15 points) Assuming the Perron-Frobenius Theorem, show that

$$\lim_{\ell \to \infty} p_k(\ell) = d_k/D.$$

(5 points) If $G$ is connected, regular, and bipartite, show that $\mathbf{M}(G)$ has an eigenvalue of $-1$. Why is $-1$ not an eigenvalue of $\mathbf{M}(G)$ if $G$ is regular, connected, but not bipartite?

3) (5 points) Draw Hasse diagrams of the 16 nonisomorphic four-element posets.

*(Bonus 5 points)*: Draw also the 63 five-element posets)

(10 points) Draw the Hasse diagram for the poset of nonisomorphic simple graphs with 4 vertices (with the subgraph ordering). What is the size of the largest antichain? How may antichains have this size?

4) Let $P$ be a finite poset and $f : P \to P$ an order-preserving bijection. I.e. $f$ is a bijection (one-to-one and onto), and if $x \le y$ in $P$ then $f(x) \le f(y)$.

(10 points) Show that $f$ is an automorphism of $P$, i.e. $f^{-1}$ is order-preserving.

(5 points) Show that the last statement is false if we do not assume that $P$ is finite.

5) We say that a graph $G$ is **regular** if all vertices of $G$ have the same degree;

$G$ is known as **integral** if its adjacency matrix has only integral eigenvalues;

and $\overline{G}$, the complement of simple graph $G$, is the simple graph such that $V(\overline{G}) = V(G)$ and $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$.

(5 points) Show that if $G$ is a simple and regular integral graph, then so is $\overline{G}$.

The **disconnected union** of $G$ and $H$, denoted as $G \sqcup H$, has vertex set $V(G \sqcup H) = V(G) \cup V(H)$ and edge set $E(G \sqcup H) = E(G) \cup E(H)$.

(5 points) Show that if $G$ and $H$ are integral graphs, then so is $G \sqcup H$.

We say that a graph $S_{n+1}$ is an $n$-**star** if the vertices are in bijection with $\{0, 1, 2, \ldots, n\}$ and the edges of $S_{n+1}$ are of the form $\Big\{ \{0, 1\}, \{0, 2\}, \{0, 3\} \ldots, \{0, n\} \Big\}$.

(10 points) For what $n \ge 1$ is $S_{n+1}$ an integral graph?

**Bonus Problems:**

We say that a graph $Z_n$ is an $n$-**cycle** if the vertices are in bijection with $\{1, 2, \ldots, n\}$ and $Z_n$ has $n$ edges: $\Big\{ \{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\} \Big\}$.

(Notice that $Z_1$ and $Z_2$ are not simple graphs.)

*(Bonus 5 points)* For what $n \geq 2$ is $Z_n$ an integral graph?

> (**Hint:** The eigenvalues of $A(Z_n)$ will involve trigonometric
>
> quantities or roots of unity.)


*(Bonus 5 points)* There are 8 connected and simple integral graphs with at most 5 vertices. Find these and give brief reasoning as to why each has integral eigenvalues.

> (**Hint:** Seven out of eight of these are familiar graphs.

*(Bonus 5 points)* Find the six connected and simple integral graphs with 6 vertices, and give brief reasoning as to why each has integral eigenvalues.

> (**Hint:** Five out of six of these 6-vertex integral graphs are regular and
>
> the other 6-vertex integral graph is a tree.)

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