Course 18.312: Algebraic Combinatorics

Homework # 9

Due Tax Day, Wednesday April 15, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) Let $f(n)$ be the number of graphs G on the vertex set $\{1, 2, \ldots, n\}$ such that every connected component of G is isomorphic to a path P_i with $i \geq 1$ vertices. (P_1 is a single vertex and P_2 is a single edge.) Set $f(0) = 1$.

   (10 points) Find $E_f(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$. (For full credit, your final answer should not contain any infinite sums.)

2) (5 points) a) Write $\frac{a+bx}{c+dx}$ as a formal power series $F(x) = a_1x + a_2x^2 + a_3x^3 + \ldots$. In other words, write a closed formula for a_i.

   Errata: You may let $a = 0$, so you can proceed with the problem as stated. For more general a, use a change of variables to write $\frac{a+bx}{c+dx} = a_1y + a_2y^2 + a_3y^3 + \ldots$ where y is a linear shift of x.

   (10 points) b) If we let $F^{\langle -1\rangle}(x) = b_1x + b_2x^2 + b_3x^3 + \ldots$, use Lagrange inversion to give a closed formula for b_i.

   (5 points) c) Using part (b) or otherwise, what is a closed form expression for $F^{\langle -1\rangle}(x)$?

3) Let $L = L(K_{rs})$ be the Laplacian matrix of the complete bipartite graph K_{rs}.

   (5 points) a) Find a simple upper bound on $rank(L - rI)$. Deduce a lower bound on the number of eigenvalues of L equal to r.

   (5 points) b) Assume $r \neq s$, and do the same as (a) for s instead of r.

   (5 points) c) Find the remaining eigenvalues of L.

(**Hint:** Use the fact that the rows of L sum to 0 and compute the trace of L.)

(5 points) d) Use (a)-(c) to compute $\kappa(K_{rs})$, the number of spanning trees of K_{rs}.

(**Bonus 5 points**) e) Give a combinatorial proof of the formula for $\kappa(K_{rs})$.