Course 18.312: Algebraic Combinatorics

Lecture Notes # 32-36 Addendum by Gregg Musiker

May 1st - 11th, 2009

1 Perfect Matchings and Domino Tilings

Definiton. A **Perfect Matching** of a graph $G = (V, E)$ is a set $M \subset E$ of distringuished edges such that each vertex $v \in V$ is incident to exactly one edge of M.

Notice that in particular that this implies that $|M| = |V|/2$ and only graphs with an even number of vertices can contain a perfect matching. Perfect matching enumeration can in general be a difficult problem but we consider the case of **bipartite planar** graphs where their number is easier to compute.

For such graphs, it is also possible to view a perfect matching of a graph as a tiling by dominoes. In particular, let G^* be the dual graph to G, which means that a vertex of G^* corresponds to a face (bounded or unbounded) region of G and vice-versa. Edges also correspond to one another. In the case that G embeds inside the $\mathbb{Z} \times \mathbb{Z}$ lattice, a domino covers two faces of G^* and perfect matchings of G exactly correspond to a domino tiling of G^*. See Kuo's paper and the reference on Aztec diamonds for typical examples of such dual graphs. These references can be found at the bottom of these lecture notes.

2 Pfaffians and Matching Enumeration

We enumerate perfect matchings by computing Pfaffians, which turn out to be determinants in the case of certain planar graphs. We warn the reader that today's lecture notes are more of a sketch than previous lecture notes. We begin with a few definitions.

Given an edge (v_i, v_j) in graph G and an orientation σ on G, we let $a_{i,j} = +1$ if $v_i \to v_j$, $a_{i,j} = -1$ if $v_j \to v_i$ and $a_{i,j} = 0$ otherwise. Given a perfect matching

$M = \{(i_1, j_1), \ldots, (i_{n/2}, j_{n/2})\}$ (where $i_r < j_r$ by convention), we let the **sign** of the matching be

$$\mathrm{sgn}(M) = (-1)^{\#\text{ crossings in the nesting pattern of } M}$$

where we draw $1, 2$ through n in a line, and the nesting pattern connects i_r and j_r for each edge of M.

Definition. An orientation σ is a **Pfaffian orientation** of G if and only if for all perfect matchings M of G, we have that each expression $\left(\prod_{r=1}^{n/2} a_{i_r, j_r}\right) \mathrm{sgn}(M)$ has the same sign, regardless of the choice of M.

Theorem. Let G be a planar simple finite graph with no bridges (i.e. no edges whose removal would disconnect the set of vertices of G). Let σ be an orientation such that the number of clockwise edges about any bounded face is **odd**. Then σ is **Pfaffian**.

To sketch the proof of this Theorem, we will use a sign-preserving map between pairs of matchings of the line $\{1, 2, \ldots, n\}$ and permutations of S_n:

$$\phi : (M_1, M_2) \longrightarrow \sigma$$

such that $\mathrm{sgn}(M_1)a_{M_1} \cdot\ \mathrm{sgn}(M_2)a_{M_2} = \mathrm{sgn}(\sigma)a_\sigma$, where σ denotes the matching $\{1, \sigma(1)\}, \{2, \sigma(2)\}, \ldots, \{n, \sigma(n)\}$. Recall that the sign of a permutation is $(-1)^{\#\text{ cycles of even length}}$.

In particular, we draw the line from 1 to n and then connect vertices on the top according to matching M_1 and connect vertices on the bottom using matching M_2. We then make cycles starting with the smallest element.

Example. Let $M_1 = \{1, 2\}, \{3, 6\}, \{4, 5\}$, $M_2 = \{1, 4\}, \{2, 5\}, \{3, 6\}$. These are both matchings of the 2-by-3 grid graph with vertices $1, 2, 3$ on top and $4, 5, 6$ in the second row. The corresponding permutation is $\sigma = (1254)(36)$.

Claim. Any σ constructed in this way only has cycles of even length.

In fact, we obtain a bijection since the process is reversible by alternating assignments of edges in the cycles to M_1 and M_2. For example, for the permutation $\sigma = (1, 3)(2, 6, 5, 8)(4, 10, 9, 7)$, we obtain

$$M_1 = \{1, 3\}, \{2, 6\}, \{5, 8\}, \{4, 10\}, \{9, 7\} \text{ and } M_2 = \{3, 1\}, \{6, 5\}, \{8, 2\}, \{10, 9\}, \{7, 4\}.$$

Furthermore, we can pull apart cycles by eliminating two crossings. This is equivalent to rearranging numbers on the line so that if i, j are in different cycles, then their placements do not affect the sign. The sign is only affected by the ordering within individual cycles, and a self-crossing can be eliminating by applying a transposition.

Getting back to the main proof, if we assume that an orientation of G is **not Pfaffian**, then there exists a pair of matchings (M_1, M_2) so that the signs of M_1 and M_2 differ:

$\operatorname{sgn}(M_1)a_{M_1} \neq \operatorname{sgn}(M_2)a_{M_2}$ which would imply by the theorem that $a_\sigma \operatorname{sgn}(\sigma) \neq +1$.

Thus it suffices to show that this sign is -1 instead.

Given an orientation with an odd number of clockwise edges around each bounded face, we build a permutation σ corresponding to a pair of matchings M_1 and M_2. Then σ breaks into a product of cycles, each of which must have even length. It follows that $\operatorname{sgn}(\sigma) = (-1)^{\#\ cycles\ of\ even\ length}$ which decomposes as $\sigma = \sigma_1 \cdots \sigma_t$. However, cycles of σ decmposes V into edges and even cycles (cycles not nec. contained on a single face) so $a_{\sigma_i} = \#$ edges oriented clockwise on face induced by cycle σ_i. So if each face has an odd number of clockise edges, then we have the result.

Lemma. Let $G = (V, E, F)$ be a simple connected planar graph without bridges with an odd number of clockwise edges on each face. Let C be a directed cycle. Then the number of edges oriented clockwise in C has the opposite parity as the number of vertices in the interior of C.

Proof omitted, but follows by application of Euler's formula applied to the subgraph of G consisting of the interior of C together with cycle C.

Thus we have shown the Theorem that an orientation of G with each face having an odd number of clockwise edges **is Pfaffian**. We will not show it in these notes, but it is possible by induction to show that such an orientation exists.

Main Theorem. If σ is a Pfaffian orientation (which we know exists in certain cases and we can inductively construct such an orientation) then we have the formula

$$\det A(G) = (\#\text{ perfect matchings of } G)^2,$$

where $\det A(G) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)a_\sigma$. The matrix $A(G)$ is the signed adjacency matrix for the oriented graph.

Application. Consider the m-by-n grid graph $L_{m,n}$ with m rows and n columns, for $m \cdot n$ vertices in all. We give this graph an orientation so that all vertical edges are oriented downwards, the horizontal of the odd (i.e. first, third, etc.) labeled rows are oriented rightwards, while the even labeled rows are oriented leftwards. Let B be the n-by-n matrix with a superdiagonal of $+1$'s, a subdiagonal of -1's and zeros everywhere else. The signed adjancency matrix for $L_{m,n}$ is the (mn)-by-(mn) block matrix obtained by taking an m-by-m matrix whose superdiagonal are the n-by-n identity matrices I_n, subdiagonal entries are $-I_n$ and diagonal entries are the B's defined above. Then all other entries are zero.

By the Theorem, $M(L_{m,n})$, the number of perfect matchings in $L_{m,n}$ is $\sqrt{\det A}$.

Corollary. For n even,

$$M(L_{m,n}) = 4^{\lfloor m/2 \rfloor} n/2 \prod_{k=1}^{\lfloor m/2 \rfloor} \prod_{\ell=1}^{n/2} \left(\cos^2(\frac{k\pi}{m+1}) + cos^2(\frac{\ell\pi}{n+1}) \right).$$

Other applications of Pffafians provide formulas for Aztec Diamonds and variants.

3 References for Height functions, Domino Tilings, and Aztec Diamonds

"Alternating sign matrices and domino tilings" by Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp

`http://arxiv.org/pdf/math/9201305v1`

and "Applications of graphical condensation for enumerating matchings and tilings" by Eric Kuo

`http://arxiv.org/pdf/math.CO/0304090.pdf`

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18.312 Algebraic Combinatorics
Spring 2009

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