Pak
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Thm (HLF)
$$|SYT(\lambda)| = n! \Big/ \prod_{x\in\lambda} h_x$$

Thm (Stanley's HCF)
$$\sum_{A\in RPP(\lambda)} t^{|A|} = \prod_{x\in\lambda} 1/1-t^{h_x}$$

Prop HCF $\Rightarrow$ HLF

Def'n $C(\lambda)$ = cone in $\mathbb{R}^n$ (i.e. set of n-tuples that satisfy all necessary inequalities given by λ)

Then $RPP(\lambda) = \{\text{integer pts in } C(\lambda)\}$

Let $\varphi(\{x_{ij}\}) = \sum_{(i,j)\in\lambda} x_{ij} = \varphi(A)$

Then $\sum_{A\in RPP(\lambda)} t^{|A|} = \sum_{A\in e(C)} t^{\varphi(A)}$

integer pts in C

Let $D = D(\lambda) = \mathbb{R}_+^n = \mathbb{R}_+\langle y_{ij}, (i,j)\in\lambda\rangle$
$\psi(A) = \psi(\{y_{ij}\}) = \sum_{ij\in\lambda} h_{ij}\, y_{ij}$

$$\sum_{B\in e(D)} t^{\psi(B)} = \underbrace{\prod_{ij\in\lambda} (1 + t^{h_{ij}} + 2 t^{h_{ij}} + \dots)}_{} = \prod_{ij\in\lambda} 1/1-t^{h_{ij}}$$

b/c can put any $n\in\mathbb{N}$ in for y_{ij}

HCF $\Rightarrow \sum_{A\in e(C)} t^{\varphi(A)} = \sum_{B\in e(D)} t^{\psi(B)}$

$|\{A\in e(C) \mid \varphi(A)\le N\}| = |\{B\in e(D) \mid \psi(B)\le N\}|$

Instead of integer pts, look at volume.

$\mathrm{vol}\,(C(\lambda)\cap \varphi(x_{ij})\le N) \sim |e(C(\lambda)\cap \varphi(x_{ij})\le N)|$

same for D + ψ

$\Rightarrow \mathrm{vol}\,(C(\lambda)\cap \varphi(x_{ij})\le N) \sim \mathrm{vol}\,(D(\lambda)\cap \psi(x_{ij})\le N)$

$\mathrm{vol}\,(\underbrace{D(\lambda)\cap \psi(x_{ij})\le N}_{\text{simplex! easy to calculate volume!}}) = 1/n!\ \prod_{ij\in\lambda} \underbrace{\ell_{ij}}_{\text{length of } ij^{th} \text{ edge}} = 1/n!\ \prod_{ij\in\lambda} N/h_{ij}$

$\mathrm{vol}\,(C(\lambda)\cap \varphi(x_{ij})\le N) = \sum_{T\in SYT(\lambda)} \mathrm{vol}\,(C(T)\cap\varphi(\,)\le N)$

b/c, let $C(T)$ be ordering of entries given by T (i.e. wherever 5 is has to be 5^{th} smallest coordinate, etc.). Then

$C(\lambda) = \bigcup C(T)$. Also $\mathrm{vol}(C(T)\cap C(T')) = 0$,

so ✓

$\mathrm{vol}\,(C(T)\cap\varphi(x_{ij})\le N) = \left|\frac{N}{n!}\right| \pm$ det't of some matrix, b/c edges are along vectors $(0\ldots 0\,x\ldots x)$

matrix is $\begin{pmatrix} \frac{1}{n} & \cdots & & \frac{1}{n} \\ 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ & & \ddots & \\ 0 & & \frac{1}{2} & \frac{1}{2} \\ & & & 1 \end{pmatrix}$

$= \frac{N^n}{n!}\cdot 1/n!$, so need

$\sum_{T\in SYT(\lambda)} \frac{N^n}{(n!)^2} = \frac{|SYT(\lambda)|}{n!}\frac{N^n}{n!} \sim \frac{N^n}{n!} 1/\prod h_{ij}$

$\Rightarrow |SYT(\lambda)| \sim n!/\prod h_{ij}$

but $\sim$ is as $N\to\infty$, so actually $=$ ✓

more in EC1 ch 7