BOUNDS ON CROSSING NUMBERS

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1. Introduction

Let G be a simple graph. The crossing number of G is the minimum number of times that the edges of G cross in any drawing of G in the plane.

First, it is important to determine exactly what is meant by a drawing.

Definition 1.1. A *drawing* of a graph is defined to be a map that sends each vertex to a point in the plane and each edge to a simple path, subject to several conditions. The endpoints of the image of an edge must be the images of its vertices. The image of a vertex must not lie in the interior of the image of any edge. If a pair of edges share an interior point, then they must cross at this point. No triple of edges can share an interior point.

In addition to the ordinary crossing number (denoted $\mathrm{cr}(G)$), several other crossing numbers are of interest.

Definition 1.2. The pairwise-crossing number $\mathrm{paircr}(G)$ is the minimum number of pairs of edges of G that cross in a drawing of G.

Definition 1.3. The odd-crossing number $\mathrm{oddcr}(G)$ is the minimum number of pairs of edges of G that cross an odd number of times in a drawing of G.

Much remains to be known about these crossing numbers, but a few relations are known. Clearly, $\mathrm{oddcr}(G) \leq \mathrm{paircr}(G) \leq \mathrm{cr}(G)$. The next section will give some bounds on $\mathrm{paircr}(G)$ in terms of the number of edges and vertices. Additional relations between $\mathrm{cr}(G)$ and the pairwise and odd crossing numbers will be presented later.

2. Bounds on crossing numbers in terms of the number of edges and vertices

Lemma 2.1. *Let $V(G)$ be the number of vertices of G, and let $E(G)$ be the number of edges. Then* $\mathrm{paircr}(G) \geq E(G) - 3V(G)$.

Suppose that G is planar. Let $F(G)$ be the number of faces in a planar drawing of G. Since G is a simple graph, each of its faces has at least three sides. The $F(G) \leq \frac{2}{3}E(G)$. By Euler's theorem, $V(G) - E(G) + F(G) = 2$, so $V(G) - \frac{1}{3}E(G) \geq 2$.

If $E(G) - 3V(G)$ is not positive, draw G such that the number of pairs of crossing edges is minimized. Removing an edge with a crossing decreases $E(G) - 3V(G)$ by one and $\mathrm{paircr}(G)$ by at least one. The lemma follows by induction.

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Theorem 2.2. *Let $V(G)$ be the number of vertices of G, and let $E(G)$ be the number of edges. If $E(G) \geq 4V(G)$, then the pairwise crossing number is at least* $\frac{E(G)^3}{64V(G)^2}$.

Draw G so that the number of pairs of crossing edges is minimized. Color each vertex either red or blue randomly and independently with probability of red p. Form the subgraph consisting of all red vertices and all edges connecting two red vertices. The expectation value for the number of vertices of this subgraph is $pV(G)$, the expectation value for the number of edges is $p^2E(G)$, and the expectation of the pairwise crossing number is at most the expectation of the number of pairs of crossing edges in the current drawing, which is $p^4\text{paircr}(G)$. By the lemma, $p^4\text{paircr}(G) \geq p^2E(G) - 3pV(G)$. Since $E(G) \geq 4V(G)$, p can be set to $\frac{4V(G)}{E(G)}$. Then $\text{paircr}(G) \geq \frac{E(G)^3}{64V(G)^2}$.

Such bounds can be used, for example, to prove the Szemerédi-Trotter Theorem.

3. Results concerning odd crossings

The following theorem was proved first by Hanani [CH34] and later by Tutte [T70].

Theorem 3.1. $\text{oddcr}(G) = 0$ *implies* $\text{cr}(G) = 0$.

János Pach and Géza Tóth [PT00] proved the following generalization, which will be used later to replace $\text{paircr}(G)$ with $\text{oddcr}(G)$ in various inequalities.

Theorem 3.2. *For any drawing of G, let G' be the subgraph of G consisting of all edges of G that cross every other edge an even number of times. Then there is a drawing of G for which the edges of G' do not cross any edge of G.*

The proof presented here is somewhat different from the original. Suppose that there was a counterexample to the theorem. Then there would be a minimal counterexample (G, G').

Lemma 3.3. *If G is a minimal counterexample, then it satisfies these properties:*

1. *No vertex of has degree one in G'.*
2. *No two vertices of degree two in G' are connected by an edge of G'.*

To prove (1), suppose that a G' had a vertex v of degree one in G'. Let e be the edge of G' that has this vertex as an endpoint. Since is (G, G') minimal, it is possible to redraw G so that e is the only edge of G' intersecting any edge of G. v can be pulled along$e$ until e no longer intersects any edges. Since e cannot intersect any edges of G', this does not introduce any new intersections of edges of G'. But this contradicts the assumption that (G, G') is a counterexample.

If G' has two vertices of degree two that share an edge, then contract that edge. Redraw so that the edges of the modified G' have no crossings. The only possible to obstacle to undoing the contraction is that the cyclic order of the edges may be incorrect. Since the contracted vertex has only two edges from G', it is possible to correct the cyclic order without introducing any crossings of those two edges. Again this contradicts the assumption that (G, G') is a counterexample. This proves the lemma.

Construct $\tilde{G}'$ as follows. Delete all vertices that have degree zero in G'. Also delete any vertex of degree two in G' and connect the two vertices that were adjacent to this vertex. By lemma 3.3, every vertex of $\tilde{G}$ has degree at least three.

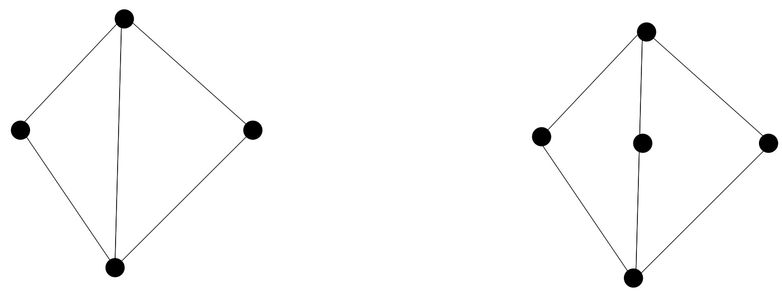

FIGURE 3.1. If every cycle of G has an empty interior or exterior, then G is one of these graphs.

Let P be the longest path of $\tilde{G}$, and suppose the sequence of vertices is $v_0, v_1, \ldots, v_n$. Because P is maximal, all neighbors of v_0 must be in the path. v_0 has at least three neighbors, so there exist $j > i > 1$ such that v_0v_i and v_0v_j are edges. So $v_0, v_1, \ldots, v_j$ is a cycle and v_0v_i is a chord of this cycle. If G consists entirely of the pullback of this cycle and v_0v_i, then it must be one of the graphs shown in 3.1.

Let C be a cycle consisting of edges in G'. The previous paragraph implies that such a cycle exists. Since C is a closed curve, it separates the plane into an "interior" and an "exterior" (possibly disconnected if C intersects itself). If G is not one of the graphs of figure 3.1, C can also be chosen so that the interior and exterior both contain at least one vertex. Define the interior graph I as follows. The vertex set is the set of all vertices of G that are either on the loop or in the interior of the loop. The edge set consists of the loop edges and all edges such that there is a segment at each end of the edge lying entirely in the interior. Note that if this is the case at one end, then it must be the case at the other since the edge must cross the loop an even number of times. Define the exterior graph E similarly. The union of I and E is G, and their intersection is the loop. Define $I' = I \cap G'$ and $E' = E \cap G$. Because (G, G') is a minimal counterexample, I and E can be redrawn to have no I' intersections and E' intersections, respectively. I claim that the redrawings can be performed so that I remains entirely inside C and E remains entirely outside C. If that is the case, then the two graphs can be glued together to form a drawing of G with no G' intersections.

Call a subgraph of G *C-connected* if it cannot be expressed as the union of two proper subgraphs having only vertices in C in common. Call the C-connected components of G (other than edges of C) *bridges.* Call the points of a bridge that are also vertices of C *anchor points.*

In order for there to be no G' crossings, each bridge must be entirely in the interior or the exterior. The first bridge of I can always be chosen to be in the interior. Now look at a second bridge of I. It is not possible for two anchor points a and b of the first bridge separate two anchor points c and d of the second bridge around C. If there is a path connecting a and b and another connecting c and d, then these paths must have crossed an odd number of times in the original drawing since they could not "go around" each other by crossing an edge of C an odd number of times. This implies that neither path can have all of its edges in G'. Make all of the G' edges of each bridge that are not contained in any G' chord of C very small. Then any intersections of bridges must be intersections between non-G' edges. The

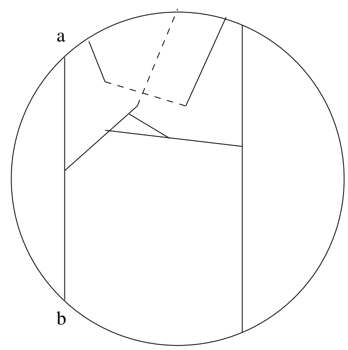

FIGURE 3.2. Possible impediments to placing a bridge in the interior, and their resolution. C is the circle, the edges of G' are solid curves, and the other edges of G are dotted curves. a and b could not have separated any other pair of vertices shown here in the original drawing. Any crossings between bridges can always be made to occur at edges not in G'.

third and subsequent bridges can be added in the same way. Therefore I can be redrawn entirely in the interior, and E can be drawn entirely in the exterior. The theorem is proved.

Corollary 3.4. $\operatorname{paircr}(G)$ *can be replaced with* $\operatorname{oddcr}(G)$ *in theorem 2.2.*

Since any graph with an odd crossing number of zero has a pairwise crossing number of zero, the argument used for theorem 2.2 works for the odd crossing number as well.

4. UPPER BOUNDS FOR $\operatorname{cr}(G)$ IN TERMS OF $\operatorname{paircr}(G)$ AND $\operatorname{oddcr}(G)$

It is not difficult to find a bound for $\operatorname{cr}(G)$ in terms of $\operatorname{paircr}(G)$.

Theorem 4.1. $\operatorname{cr}(G) \leq \binom{2\operatorname{paircr}(G)}{2}$

The following lemma is useful in proving the theorem.

Lemma 4.2. *In the drawing of G with the fewest crossings, each pair of edges of G intersect at most once.*

I will show that if a pair of edges crosses more than once, then it is possible to reduce the crossing number. There are two configurations that need to be taken into account. For each configuration, the parts of the edges between the two intersection points are swapped. The total number of crossings is decreased by two. Figure 4.1 shows the configurations and swaps. It is possible that this procedure will cause an edge to cross itself, which is not allowed by the definition of a drawing given in the introduction. However, it is not difficult to see that the self-intersections can always be eliminated, and this decreases the crossing number. This proves the lemma.

G can be drawn so that $\operatorname{paircr}(G)$ edges cross. In this drawing, $n \leq 2\operatorname{paircr}(G)$ edges cross at least one other edge. When the edges are rearranged so that $\operatorname{cr}(G)$ is minimized, the number of crossings is at most $\binom{n}{2}$.

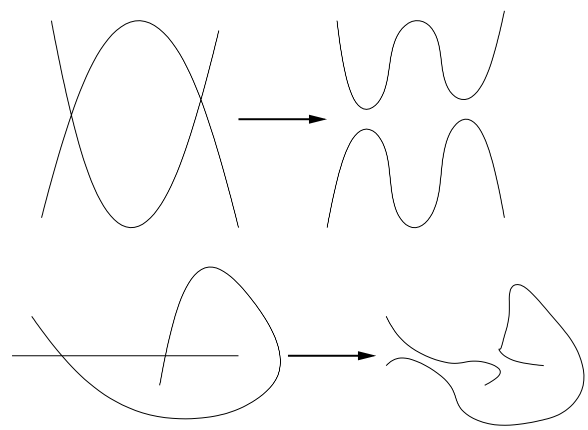

FIGURE 4.1. If a pair of edges crosses more than once, then it is always possible to reduce the crossing number. There are two possibilities, depending on whether the crossings have the same or opposite orientation.

Pach and Tóth used theorem 3.2 to replace $\operatorname{paircr}(G)$ with $\operatorname{oddcr}(G)$ in the above bound.

Theorem 4.3. $\operatorname{cr}(G) \leq \binom{2\operatorname{oddcr}(G)}{2}$

Consider a drawing of G with the minimal number of pairs of edges that cross an odd number of times. Let G' denote the subgraph consisting of the edges that cross every edge an even number of times in this drawing. Then

$$\operatorname{oddcr}(G) \geq \frac{1}{2}(E(G) - E(G')). \tag{4.1}$$

Now use theorem 3.2 to redraw the graph so that no edge of G' crosses any edge of G. Among these redrawings, choose the one with the fewest crossings.

The procedure of Lemma 4.2 never creates G' edges if none existed originally, so it can be applied again. This means that

$$\operatorname{cr}(G) \leq \binom{E(G) - E(G')}{2}. \tag{4.2}$$

Combining this with equation 4.1 yields the theorem.

Pavel Valtr [V05] improved theorem 4.1 slightly.

Theorem 4.4. $\operatorname{cr}(G) \leq O\left(\frac{\operatorname{paircr}(G)^2}{\log \operatorname{paircr}(G)}\right)$

Let $k = \operatorname{paircr}(G)$. Choose a drawing of G so that the number of pairs of crossing edges is minimized, and among such drawings, choose the one with the smallest number of crossings. Call this drawing D_0. Let t be an integer. Call an edge *light* if it crosses at least one but at most t other edges. Call an edge *heavy* if it crosses more than t other edges. Let l and h be the number of light and heavy edges, respectively. The proof will require the following lemma of M. Schaefer and D. Števankovič [SS04].

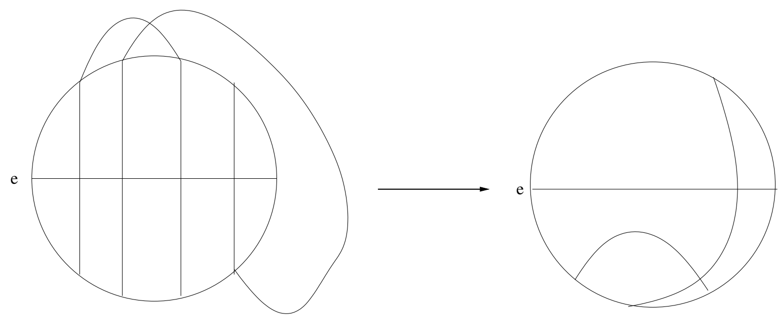

FIGURE 4.2. An example of the procedure of Schaefer and Števankovič for reducing the number of crossings without introducing any new pairs of crossing edges.

Lemma 4.5. *If an edge e crosses at most t other edges and the total number of crossings is at least 2^t, then it it is possible to redraw the graph with fewer crossings without introducing any new pairs of crossing edges.*

Consider the sequence of edges intersecting e. If one travels along e and keeps a running total of the number of times each edge has been intersected, there are 2^t possibilities mod 2. The number of running totals is one more than the number of crossings, so it exceeds 2^t. By the pigeonhole principle, some pair of running totals must be the same mod two. This means that there is a region of the curve in which e crosses every edge an even number of times. Draw a small window around this region not containing any additional crossings or endpoints. For each edge f intersecting e, chose an orientation for f and number the intersections of f with the window in order along f. The total number of intersections between f and the window is divisible by four. Call this number $4n_f$. Apply a homeomorphic map of the plane that sends the window to a disc and the segments of the f's inside the window to vertical line segments. Delete all of these segments. For each i with $0 \leq i < n_f$, the connection between $4i+2$ and $4i+3$ lies outside the window. Move these inside the window by inverting them about the circle. Then reflect them about the horizontal diameter of the circle. Now the edge f goes from 1 to 4 (inside) to 5 (outside) to 8 (inside) to 9 (outside) etc. The number of crossings of f with e is now $2n_f$, and no new crossings among edges other than e were introduced. The lemma is proved.

By the lemma, each light edge has at most $2^t - 1$ crossings. Then the number of crossings between light edges is at most $\frac{l}{2}(2^t - 1)$. Now redraw the graph so that the number of light-light crossings is minimized, the number of light-heavy crossings is minimized subject to the previous condition, and the number of heavy-heavy edges is minimized subject to the previous two conditions. Suppose that a heavy edge e crossed a light edge f at least twice. Let n_e and n_f be the number of light edges crossing e and f, respectively, in between the two crossings of e and f. If $n_f \leq n_e$, then the number of heavy-light crossings can be decreased without changing the number of light-light crossings by routing f along e and removing the two $e-f$ crossings. If $n_f > n_e$, then the number of light-light crossings can be decreased by routing e along f. Therefore a heavy edge and a light edge can cross only once. Similarly, if two heavy edges had a crossing in common, then one could be routed

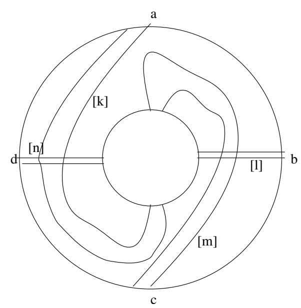

FIGURE 5.1. This construction of Pelsmajer, Schaefer, and Števankovič gives a family of graphs G with $\mathrm{oddcr}(G) \leq (\frac{\sqrt{3}}{2} + o(1))\mathrm{paircr}(G)$.

along the other so as to decrease the number of heavy-light crossings or to keep this number the same and decrease the number of heavy-heavy crossings.

Therefore the number of crossings involving heavy edges is at most $\binom{h}{2} + hl \leq h(h+l) \leq 2hk$. The number of pairs of edges that cross is at least $\frac{l}{2}$, and it is also at least $\frac{th}{2}$. Therefore $\mathrm{cr}(G) = O(\frac{k^2}{t} + 2^t k)$. Taking $t = \frac{1}{2}\log_2 k$ yields the theorem.

5. Other interesting results concerning crossing numbers

The *rectilinear crossing number* $\mathrm{lincr}(G)$ is the minimum number of drawings in a graph with straight line edges. It is clear that $\mathrm{lincr}(G) \geq \mathrm{cr}(G)$. It has also been proved that $\mathrm{lincr}(G) = \mathrm{cr}(G)$ if $\mathrm{cr}(G)$ is zero. However, as was shown by Bienstock and Dean [BD93], there are graphs with crossing number four and arbitrarily high rectilinear crossing number.

The result of Hanani and Tutte that $\mathrm{oddcr}(G) = 0$ implies $\mathrm{cr}(G) = 0$, along with the fact that each pair of edges crosses at most once in the drawing with the minimum number of crossings, suggested that that $\mathrm{oddcr}(G)$ might always be equal to $\mathrm{cr}(G)$. However, earlier this year, Pelsmajer, Schaefer, and Števankovič [PSS05] showed that this is not the case.

Theorem 5.1. *There exist graphs for which* $\mathrm{oddcr}(G) \leq (\frac{\sqrt{3}}{2} + o(1))\mathrm{paircr}(G)$.

The construction will be presented here; the proof that it works can be found in [PSS05]. Let the vertices of the graph be $a_1, \ldots, a_k$, $b_1, \ldots, b_l$, $c_1, \ldots, c_m$, $d_1, \ldots, d_n$, $a'_1, ..., a'_k$, $b'_1, \ldots, b'_l$, $c'_1, \ldots, c'_m$, $d'_1, ..., d'_n$. Each a vertex to the corresponding a' vertex, each b vertex to the corresponding b' vertex, etc. Also connect consecutive a vertices to each other, consecutive a' vertices to each other, consecutive b vertices to each other, etc. Connect a_k to b_1, b_l to c_1, c_m to d_1, and d_n to a_1. Finally, connect a'_k to d'_1, d'_n to c'_1, c'_m to b'_1, and b'_l to a'_1.

Lemma 5.2. *If $k \leq l \leq n \leq m$ and $k + n \geq m$, then for sufficiently large n the crossing number and pair crossing number are both $kn + lm$, and the odd crossing number is $ln + km$.*

See[PSS05] for a proof.

Take $l = n$, $k = \lfloor n\frac{\sqrt{3}-1}{2}\rfloor$, and $m = \lfloor n\frac{\sqrt{3}-1}{2}\rfloor$. Then the crossing number and pairwise crossing number are both $n\left(\lfloor n\frac{\sqrt{3}-1}{2}\rfloor + \lfloor n\frac{\sqrt{3}+1}{2}\rfloor\right) > \sqrt{3}n^2 - 2$, while the odd crossing number is $\lfloor n\frac{\sqrt{3}-1}{2}\rfloor\lfloor n\frac{\sqrt{3}+1}{2}\rfloor \leq \frac{3}{2}m^2$. This proves that there are graphs whose pairwise crossing number exceeds their odd crossing number. It is still not known whether the pairwise crossing number is always equal to the crossing number.

Garey and Johnson [GJ83] showed that the problem of determining whether $\mathrm{cr}(G)$ is less than an integer K is NP-complete. Pach and Tóth [PT00] proved that the same is true with $\mathrm{cr}(G)$ replaced by $\mathrm{oddcr}(G)$. They also showed that the problem of determining whether $\mathrm{paircr}(G) < K$ is NP-hard.

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