18.330 :: Homework 3 :: Spring 2012 :: Due March 15

1. (2.5pts) Design a one-sided second-order accurate finite difference formula to approximate $f'(0)$ from the samples $f(0), f(h)$, and $f(2h)$, by differentiating an adequately chosen interpolation polynomial. (Notice that $x = 0$ is the left endpoint.) Justify your answer either with Taylor expansions or by invoking a result seen in class.

2. (2.5pts) Consider the $[1\ 4\ 1]$ Simpson's rule seen in class: $\int_{-h}^{h} f(x)dx \simeq \frac{h}{3}(f(-h) + 4f(0) + f(h))$.

   a) Although Simpson's rule was derived from parabolas, prove that it integrates all cubic polynomials exactly.

   b) Use this result to prove that the local error of computing $\int_{-h}^{h} f(x)dx$ by Simpson's rule is $O(h^5)$ provided $f \in C^4[-h, h]$.

   (For your information: the polynomial rule obtained from integrating the cubic interpolant over 4 points yields a different rule called the 3/8 rule, which happens to also have a local error $O(h^5)$.)

3. (2.5pts) Implement clamped cubic spline interpolation with zero derivative at the endpoints. Apply it to fit a spline to the data points $\{x_j, y_j\}$ with knots $x_j = j$ for $0 \le j \le N$, and

   $$y_0 = 1, \qquad y_1 = 0, \qquad y_k = 1 \quad \text{for } k \ge 2.$$

   Use $N = 10$ and plot the result between x_0 and x_{10} (on a fine grid with many more than 10 points.)

4. (2.5pts) A quadratic spline is a piecewise quadratic interpolant, imposed to have one continuous derivative at the knots x_j. For data $\{x_j, y_j\}$, it is given in the interval $[x_j, x_{j+1}]$ by the formula

   $$s_j(x) = y_j + z_j(x - x_j) + \frac{z_{j+1} - z_j}{2(x_{j+1} - x_j)}(x - x_j)^2,$$

   where the slopes z_j obey the recurrence relation

   $$z_{j+1} = -z_j + 2\frac{y_{j+1} - y_j}{x_{j+1} - x_j}.$$

   There is only one degree of freedom that the interpolation and continuity conditions do not specify: the value of z_0. In what follows take $z_0 = 0$.

   a) Compute z_j, $1 \le j \le 10$, for the same sequence $\{x_j, y_j\}$ as considered in question 3.

   b) Plot the quadratic spline between x_0 and x_{10}.

   (c) In one sentence, explain why quadratic splines are inferior to both linear and cubic splines for practical curve fitting.