1.4 Orthogonal (Unitary) matrices

Tools of the trade:

Definition:

$$Q^* = Q^{-1} \tag{1.7}$$

i.e., $Q^*Q = I$.

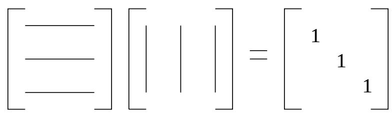

Figure 1.1: Orthogonal Matrices.

$$\langle q_i, q_i \rangle = q_i^* q_i = \delta_{ij} \tag{1.8}$$

Orthogonal matrices preserve length of a vector:

$$\|Qx\|_2 = \|x\|_2 \tag{1.9}$$

Proof:

$$\begin{aligned} \|Qx\|_2^2 &= (Qx)^* Qx \\ &= x^* Q^* Qx \\ &= x^* \underbrace{Q^{-1}Q}_{I} x \\ &= x^* x \\ &= \|x\|_2^2 \end{aligned} \tag{1.10}$$

1.5 Vector Norms

Definition: A norm $\|\cdot\|$ is a function from $\mathbb{C}^n$ into $\mathbb{R}$ such that:

1. $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $\|\alpha x\| = |\alpha| \cdot \|x\|$

Example 1: $1, 2, \infty$ norms:

$$\|x\|_1 = \sum |x_i| \quad (1.11)$$

$$\|x\|_2 = \sqrt{\sum |x_i|^2} \quad (1.12)$$

$$\|x\|_\infty = \max_i |x_i| \quad (1.13)$$

Example 2: Weighted norms:

$$W = \text{diag}(x_i) \Rightarrow \|x\|_w = \sqrt{\sum |x_i w_i|^2} \quad (1.14)$$

1.6 Matrix Norms

Same as vector norms, functions from $\mathbb{C}^{n\times n}$ into $\mathbb{R}$, satisfying:

1. $\|A\| \geq 0$
2. $\|A + B\| \leq \|A\| + \|B\|$
3. $\|\alpha A\| = |\alpha| \cdot \|A\|$

Some matrix norms are more useful: Induced Matrix Norms

$$\|A\| = \sup_x \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| \quad (1.15)$$

$1, 2, \infty$ norms for matrices:

$$\|A\|_1 = \max_i \sum_j |a_{ij}| \qquad \text{max row sum} \quad (1.16)$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^* A)} \quad (1.17)$$

$$\|A\|_\infty = \max_j \sum_i |a_{ij}| \qquad \text{max col sum} \quad (1.18)$$

They satisfy $\|AB\| \leq \|A\| \cdot \|B\|$, which is not satisfied by all matrix norms, but it is by the ones induced by vector norms and the Frobenius norm:

$$\|A\|_F^2 = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} = \sqrt{\text{tr}(A^* A)} \quad (1.19)$$

1.7 Invariance under Unitary Multiplication

Proposition: If Q is unitary ($Q^*Q = I$), then

$$\|QA\|_2 = \|A\|_2 \tag{1.20}$$

and

$$\|QA\|_F = \|A\|_F \,. \tag{1.21}$$

Proof: Let x: $\|QAx\|_2 = \|QA\|_2$

$$\|QA\|_2 = \|Q(Ax)\|_2 = \|Ax\|_2 \leq \|A\|_2 \tag{1.22}$$

Also if y: $\|Ay\|_2 = \|A\|$, then

$$\|A\|_2 = \|Ay\|_2 = \|QAy\|_2 \leq \|QA\|_2 \tag{1.23}$$

thus

$$\|A\|_2 = \|QA\|_2 \,. \tag{1.24}$$

$$\|A\|_F = \sqrt{\operatorname{tr}(A^*A)} = \sqrt{\operatorname{tr}((QA)^*QA)} = \|QA\|_F \tag{1.25}$$