Operator Splitting

IVP: $u_t = Au + Bu$

where A, B differential operators.

Most accurate: Discretize $Au + Bu$, and time step with high order.
But: Sometimes not possible, or too costly.

Alternative: fractional steps

$$\begin{cases} \text{Time step} & t \to t + \Delta t : \\ \text{(1) Solve} & u_t = Au \\ \text{(2) Solve} & u_t = Bu \end{cases}$$

Special case: A,B linear

Solution operators:

$e^{tA}u_0$: solution of $u_t = Au, u(0) = u_0$
$e^{tB}u_0$: solution of $u_t = Bu, u(0) = u_0$
$e^{t(A+B)}u_0$: solution of $u_t = Au + Bu, u(0) = u_0$

True solution: $u(t + \Delta t) = e^{\Delta t(A+B)}u(t)$
Lie splitting: $u_L(t + \Delta t) = e^{\Delta tA}e^{\Delta tB}u(t)$
Strang splitting: $u_S(t + \Delta t) = e^{\frac{1}{2}\Delta tA}e^{\Delta tB}e^{\frac{1}{2}\Delta tA}u(t)$
SWSS splitting: $u_{SW}(t + \Delta t) = \frac{1}{2}(e^{\Delta tA}e^{\Delta tB} + e^{\Delta tB}e^{\Delta tA})u(t)$

Local Truncation Errors:

Lie: $u_L(t + \Delta t) - u(t + \Delta t) = \frac{\Delta t^2}{2}[A, B]u(t) + O(\Delta t^3)$
Strang: $u_s(t + \Delta t) - u(t + \Delta t) = \Delta t^3 \cdot (\frac{1}{12}[B, [B, A]] - \frac{1}{24}[A, [A, B]])u(t) + O(\Delta t^4)$
SWSS: $u_{sw}(t + \Delta t) - u(t + \Delta t) = O(\Delta t^3)$

Commutator: $[A, B] = AB - BA$

> If operators A and B commute, then all splittings are exact;
> Otherwise:
> - Lie (globally) first order accurate,
> - Strang and SWSS (globally) second order accurate.

Ex.: Convection-Diffusion equation

$u_t + cu_x = du_{xx}$
Solution: $u(x, t) = h(x - ct, t)$, where h solves $h_t = dh_{xx}$
$Au = -cu_x$ and $Bu = du_{xx}$
$ABu = -c(du_{xx})_x = -cdu_{xxx} = BAu \Rightarrow [A, B] = 0$
$\Rightarrow$ splitting exact $\Rightarrow$ use Lie splitting

Ex.: Convection-Reaction equation

$u_t + cu_x = a - bu$
$Au = -cu_x$ and $Bu = a - bu$
$ABu = -c(a - bu)_x = bcu_x$
$BAu = a - b(-cu_x) = a + bcu_x$
$[A, B] \neq 0 \Rightarrow$ use Strang splitting to be second order accurate

Ex.: Dimensional splitting

2D Advection
$u_t + au_x + bu_y = 0$
(1) $u_t + au_x = 0$ for Δt
(2) $u_t + bu_y = 0$ for Δt
$Au = -au_x, Bu = -bu_y, [A, B] = 0$

Remark: No error due to splitting, if
$u_t = Au$ and $u_t = Bu$ solved exactly.

If discretized in time, results will in general differ.

Ex.: FE unsplit: $\dfrac{U_{ij}^{n+1} - U_{ij}^n}{\Delta t} = -a\dfrac{U_{ij}^n - U_{i-1,j}^n}{\Delta x} - b\dfrac{U_{ij}^n - U_{i,j-1}^n}{\Delta y} \quad [a, b > 0]$

Lie-splitting: (1) $\dfrac{U_{ij}^* - U_{ij}^n}{\Delta t} = -a\dfrac{U_{ij}^n - U_{i-1,j}^n}{\Delta x}$

(2) $\dfrac{U_{ij}^{n+1} - U_{ij}^*}{\Delta t} = -b\dfrac{U_{ij}^* - U_{i,j-1}^*}{\Delta y}$

Hence:

$$\begin{aligned} U_{ij}^{n+1} =& U_{ij}^n - \frac{a\Delta t}{\Delta x}(U_{ij}^n - U_{i-1,j}^n) - \frac{b\Delta t}{\Delta y}(U_{ij}^n - U_{i,j-1}^n) \\ &+ \frac{ab(\Delta t)^2}{\Delta x \Delta y}(U_{ij}^n - U_{i-1,j}^n - U_{i,j-1}^n + U_{ij}) \end{aligned}$$

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