

2.050J/12.006J/18.353J Nonlinear Dynamics I: Chaos, Fall 2012

Problem Set 4

Due at 12:01 pm on Friday, October 5th, in the box provided. No late psets are accepted. If you collaborated with other students in the class, list their names on the title sheet. The work that you submit must be your own.

Main concepts: Three bifurcations in 1D dynamical systems (saddle-node, transcritical, and pitchfork).

Reading: Strogatz Ch. 3.

Problem 1: Bifurcations of 1D dynamical systems. Normal forms.

For each of the following problems,

- sketch all the qualitatively different vector fields that occur as r is varied.
- Find at which x and for which critical values of the parameter r the bifurcation occurs.
- Which bifurcation is it (saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork)?
- Finally, sketch or plot in MATLAB the bifurcation diagram of fixed points x^* versus r , and mark the stable branches as solid lines, while the unstable branches as dashed lines. (Note: for some of the cases it is simpler to plot the bifurcation in MATLAB).

(a) $\dot{x} = r - \cosh x$

(b) $\dot{x} = xr - xe^x$

(c) $\dot{x} = rx - \sinh x$

(d) $\dot{x} = 5 - re^{-x^2}$

(e) $\dot{x} = rx + \frac{x^3}{1+x^2}$

(f) $\dot{x} = r - \cos x$

Problem 2: Hierarchy of models of fishery

(based on Strogatz 3.7.3 and 3.7.4)

To understand the underlying dynamics of a real-world problem, one frequently considers a hierarchy of models – from the simplest one to a more complicated one. In this problem we consider three models of fisheries of increasing complexity. Your task is to analyze them using the material from the class, and to see if one model is better than the other (for example, the dynamics of which one is more physical).

The simplest model for fishery would be that in the absence of fishing the population of fish is assumed to grow logistically $\dot{N} = rN(1 - N/K)$. Then to model the fishing rate the simplest choices are:

(a) $\dot{N} = rN(1 - N/K) - H$, where the fish is harvested at a constant rate H ,

(b) $\dot{N} = rN(1 - N/K) - HN$,

(c) $\dot{N} = rN(1 - N/K) - H\frac{N}{A+N}$,

1. Interpret physically the form of the harvesting rate in (b) and (c). What does the parameter A stand for physically. Which model of a harvesting rate do you think is more physical?
2. Non-dimensionalize the models to obtain the forms
 - a) $\dot{x} = x(1 - x) - h$
 - b) $\dot{x} = x(1 - x) - hx$
 - c) $\dot{x} = x(1 - x) - h\frac{x}{a+x}$
3. For models (a) and (b) find the fixed points and determine their stability for different values of h . Are any of them unphysical?
4. (MATLAB) plot the bifurcation diagrams for the cases (a) and (b). Label stable, unstable branches and bifurcation points.
5. Do the normal form analysis in the cases (a) and (b) and obtain analytically the results you got using MATLAB in the item above.

For model (c):

6. Show that the system can have one, two, or three fixed points, depending on the values of a and h . Classify the stability of the fixed points in each case.
7. Analyze the dynamics near $x = 0$ and show that the bifurcation occurs when $h = a$. What type of bifurcation is it?
8. Show that another bifurcation occurs when $h = \frac{1}{4}(a + 1)^2$, for $a < a_c$, where a_c is to be determined. Classify the bifurcation.
9. Plot the stability diagram of the system in (a, h) parameter space. (The stability diagrams in 2-dimensional parameter space will be shown in class on Tuesday). Can hysteresis occur in any of the stability regions?

And finally, your conclusions:

10. Now that you know the dynamics of the three models and its dependence on parameters, conclude which model is more realistic, and which important (or not) physical effects it takes into account. Which, if any, important effects does it miss?

Matlab help:

There is an easy way to plot parametrized curves in MATLAB which, once you see it, is obvious. It is very useful for plotting bifurcation curves.

Usually one is plotting a curve, for example

```
x = 0:0.1:10; plot(x,sin(x));
```

However, say you need to plot a bifurcation diagram for $\dot{x} = r + \sin(x)$, where r is a parameter, on the (r, x) plane. Then you can reverse the arguments in `plot`:

```
x = -2:0.2:2; plot(-sin(x),x);
```

In general, when given a parameterized curve $x(\theta) = \cos(\theta)$, $y(\theta) = \sin(\theta)$, to plot it in MATLAB do

```
theta = [0:0.1:2*pi]; x = cos(theta); y = sin(theta); plot(x,y); axis equal;
```

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