

2.050J/12.006J/18.353J Nonlinear Dynamics I: Chaos, Fall 2012

Problem Set 7

Due at 12:01 pm on Friday, November 2nd, in the box provided. No late psets are accepted. If you collaborated with other students in the class, list their names on the title sheet. The work that you submit must be your own.

Main concepts: Poincaré-Bendixon theorem, trapping region, Hopf bifurcation

Reading: Strogatz Ch. 7.3, 8.2-3

Problem 1: Poincaré-Bendixon theorem, constructing a trapping region

Consider the system

$$\dot{x} = -\mu x - y + x^3, \dot{y} = x - \mu y + y^3.$$

- (a) Show that there exists a periodic orbit for $\mu = 1$ by constructing a trapping region
- (b) (MATLAB) Plot the phase portrait for $\mu = 1$ and $\mu = -1$
- (c) Find the value of μ for which the Hopf bifurcation happens.

Problem 2: A circular limit cycle

Consider a nonlinear oscillator:

$$\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0.$$

- (a) Find and classify all fixed points (in the phase plane (x, \dot{x})) for $a > 0$, $a < 0$ and $a = 0$.
- (b) Plot the phase portrait for $a = 1$ and $a = -1$. (After plotting them in MATLAB write the command `axis equal`; It scales the x and the y axis the same, so that your circular orbits look circular, and not ellipsoidal.) The phase portraits look strikingly similar, what is the transformation to change one of them into the other one?
- (c) Show analytically that there is a limit cycle for $a \neq 0$. What is its stability for $a > 0$ and $a < 0$?
- (d) Determine the radius of the limit cycle analytically.
- (e) Determine the period of the limit cycle analytically; check the answer with your numerics.
- (f) Check that the condition for the Hopf bifurcation for the eigenvalues near $a = 0$ is satisfied, is the bifurcation supercritical, subcritical or degenerate?

Problem 3: Oscillating chemical reaction

Strogatz 8.3.2

Problem 4: Numerics versus a Liapunov function

For this system a Hopf bifurcation occurs at $\mu = 0$,

$$\begin{aligned}\dot{x} &= \mu x + y \\ \dot{y} &= -x + \mu y - x^2 y.\end{aligned}$$

- (a) MATLAB: Plot the phase portraits for some $\mu < 0$ and $\mu > 0$. Deduce if the bifurcation is supercritical or subcritical. (Note: use `ode45`, since `streamslice` plots will be inconclusive).
- (b) For $\mu = 0$ check analytically if the origin is stable or not, does it give you the same prediction as numerics? (Hint: for $\mu = 0$ the system can be easily rewritten as a second-order equation for x which models a nonlinear oscillator, this will help you to construct a Liapunov function).
- (c) Sketch a bifurcation diagram.

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