

# 2.050J/12.006J/18.353J Nonlinear Dynamics I: Chaos, Fall 2012

## Midterm Practice Problems

The midterm exam will have a classroom written portion and a take-home computational portion.

The best preparation for the exam is problem sets (the solutions are posted online). The solution to the PSet 6 will be online on Friday just after it is due.

### Topics

You now know how to analyze models of physical phenomena that are modeled as one and two-dimensional systems of ODEs that can depend on parameter. The steps in the study are

1. Non-dimensionalize the system, thus reducing the number of parameters in the problem.
2. Find fixed points, study the local phase portrait near these fixed points
  - (a) if the fixed points are hyperbolic, then the linear theory gives the correct phase portrait
  - (b) if the linear system predicts centers, then one has to do extra work. Nonlinear centers appear in the cases of conservative systems, reversible systems. Otherwise more work is needed to determine if the phase portrait is a spiral, or a center, or something else. (Strogatz Ch. 7).
  - (c) in case of zero eigenvalues one has to use different techniques (for example, graphical techniques) to deduce the stability of a fixed point. For example  $\dot{x} = x^2$  ( $x = 0$  is semi-stable),  $\dot{x} = x^3$  ( $x = 0$  is unstable), etc.
3. Deduce some of the characteristics of the global phase portrait (look at nullclines, "fences and funnels" techniques from 18.03, perhaps you can find a trapping region).
4. Check if there could be closed orbits in the system, or you can rule out their existence
  - (a) in case of Liénard systems, there exists a unique stable limit cycle
  - (b) check the index theory, perhaps you can show that there are no closed orbits
  - (c) check if the system is conservative (then in 2D you can only have saddles and centers)
  - (d) check if you can rule out closed orbits completely (Dulac's criterion, existence of a Liapunov function, gradient systems)
5. Study dependence of the system on parameters – bifurcation theory.
  - (a) Bifurcations we studied in both 1D and 2D are saddle-node, transcritical and pitchfork bifurcations (but in PSets you had examples of others).
  - (b) Find where they occur (in 1D for  $\dot{x} = f(x, r)$ ,  $r$ -parameter, can look at the system  $f(x, r) = 0$ ,  $\partial f(x, r)/\partial x = 0$ ).
  - (c) Expand  $f(x, r)$  near a critical point  $x = x^*$  (fixed point) and  $r = r_c$  (where the bifurcation happens), find the normal form, deduce the type of the bifurcation

(d) Sometimes it is easier to see graphically when a bifurcation occurs. For example, if  $\dot{x} = f(x) = e^{-x} - (x + a)$ , then the bifurcation occurs when the curve  $y = e^{-x}$  touches the straight line  $y = x + a$ . The same graphical technique is applicable to the second order systems when you can check when the nullclines start to intersect as you vary a parameter.

6. Last but probably the most important item: interpret your answer physically. For example, in the laser system you find a transcritical bifurcation, and that a stable fixed point at zero loses its stability at the bifurcation point and a new stable equilibrium exists for the larger value of parameter. However, you should reinterpret your answer physically that below the critical value of the parameter the system acts like a lamp, while above the "laser threshold" it acts like a laser.

## Sample problems

1. Non-dimensionalization:

Pset 2, 3, 6

Extra sample problems: 3.7.5, 6.4.4-6, 6.4.7, 8.1.10 (these are all long problems, they all have good examples of non-dimensionalization, but if you try to fully complete all the questions in them, it will take a very long time)

2. Stability of fixed points, linear phase portraits:

all psets. Special cases: Pset 2, problem 1.

Extra sample problems:

$$\dot{x} = x^3, \dot{y} = y$$

$$\dot{x} = -x^2, \dot{y} = -y$$

Strogatz problems: 5.2.3 - 5.2.10.

3. Nonlinear centers:

Pst 2, problem 5, Pset 3 (conservative systems), reversible systems (Strogatz, problems in 6.6).

Extra sample problems: 6.5.13

4. Using nullclines to deduce something about the system:

in case of bifurcations (Strogatz, example 8.1.1.), in case of a trapping region (Strogatz, Glycolysis example 7.3.2).

Extra sample problems:

8.1.6,

5. Closed orbits:

(1. prove that there are some) NL centers (pset 3), 6.5.3, 6.6.5, 6.1.1

(2. rule out closed orbits): Pset 6, 6.8.7, 7.2.6, 7.2.8, 7.2.12, 7.2.14

(3. limit cycles): Pset 6, 7.4.2, 7.1.7

(4. non-isolated periodic orbits): everything about linear and nonlinear centers

6. Closed orbits: limit cycle versus centers:

Pet 6, problem 1 (case  $\varepsilon = 0$  versus  $\varepsilon > 0$ ).

Extra sample problems:

7. Bifurcations in 1D, 1 parameter:

Pset 4,

Extra sample problems: 3.1.1-4, 3.2.1-4, 3.4.1-4, 3.4.5-10, 3.1.13. Plot bifurcation diagrams. 3.5.4.

8. Bifurcations in 1D, 2 parameters:

Pset 4,

Extra sample problems: 3.7.5

9. Bifurcations in 2D:  
 PSet 5, problem 1  
 Extra sample problems: 8.1.1-7, 8.1.10, 8.1.13
10. Index theory:  
 Pset 5  
 Extra sample problems: 6.8.2-5 (unusual examples), 6.8.9, 6.8.6,  
 Can there be two fixed points inside a limit cycle? If so, give an example (graphically, do not make up equations for it).
11. Oscillators:  
 linear oscillators: (18.03) underdamped, undamped, critically damped oscillators (Pset 1),  
 nonlinear oscillators:  $\ddot{x} + b\dot{x} + \sin x = 0$ , Van der Pol, ... (Pset 6, first lectures).

## Matlab

For the take-home section, you should be able to plot things in Matlab and be able to use it to make some research conclusions. Sample problems are of the type:

1. Use `streamslice` to plot the phase portrait for any of the 2D systems listed above
2. Use `plot` to plot bifurcation diagrams and trajectories  $x(t)$ ,  $y(t)$  as a function of time
3. For many problems `streamslice` does not give good results at all (examples in the last Matlab primer). Use `ode45` in those cases, and choose where you start the trajectories yourself.

You should be able to plot phase portraits for any one of the problems listed above and in the psets. Make sure that your plots are *\*not\** inconclusive. For example:

1. if you want to show that a trajectory converges to a fixed point, and you compute the trajectory via `ode45`, choose a long enough time-interval to show that it does.
2. if you want to show that trajectories are attracted to a fixed point from all directions, pick the initial conditions all around the fixed point, and not in one small area to one side of the fixed point
3. label your plots accordingly. It is useful to put the parameter value in the title when you vary the parameter and have many plots. In the following example we vary  $\mu$  between 0 and 1 in steps of 0.1, then plotting something with the title reading  $\mu = 0.3$ :
4. We are dealing with solutions to ODEs. They are all differentiable. Therefore in your plots the solutions should never have any corners (which happen if you plot a function with low resolution, for example `x = [0:1:10]; plot(x,sin(x))`); So always make sure that you have enough resolution when you plot the trajectories (in phase space, or time-evolution plots).

```
mu = [0:0.1:1]; x = [0:0.1:3];
for i = 1:length(mu),
figure(1);
plot(x,sin(x)*mu(i));
title(['mu = ',num2str(mu(i))]);
end
```

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