18.366 Random Walks and Diffusion, Spring 2005, M. Z. Bazant.

Problem Set 1

Due at lecture on Th Sept 21.

1. **Rayleigh's walk.** Consider an isotropic random walk in three dimensions with IID displacements of length, a, given by the PDF,

$$p(\vec{x}) = \frac{\delta(r-a)}{4\pi a^2} \qquad (r = |\vec{x}|)$$

   (a) Derive the following formula for the PDF of the position after N steps,

$$P_N(\vec{x}) = \frac{1}{2\pi^2 r} \int_0^\infty u \, \sin(ur) \left[\frac{\sin(ua)}{ua}\right]^N du.$$

   (b) Using Laplace's method, derive the asymptotic formula,

$$P_N(\vec{x}) \sim \left(\frac{3}{2\pi a^2 N}\right)^{3/2} \exp\left(-\frac{3r^2}{2a^2 N}\right) \tag{1}$$

   as $N \to \infty$ for $r = O(\sqrt{N})$, consistent with the Central Limit Theorem.

   (c) *Extra Credit:* Derive the next term in the Gram-Charlier expansion of $P_N(\vec{x})$, and show that 'central region' where Eq. (1) holds is much wider, $r = O(N^{3/4})$.

2. **Cauchy's walk.** Consider a random walk in one dimension with independent, *non*-identical displacements given by Cauchy's PDF,

$$p_n(x) = \frac{A_n}{a_n^2 + x^2}$$

   for some positive sequence $\{a_n\}$.

   (a) Derive the characteristic function, $\hat{p}_n(k)$, and determine A_n. Explain why $\hat{p}_n(k)$ is not analytic at the origin.

   (b) Derive the PDF of the position after N steps, $P_N(x)$. For $a_n = a =$constant, how does the half-width of $P_N(x)$ scale with N? Explain how your result is consistent with the Central Limit Theorem.

3. **Ergodicity breaking.** Perform simulations of the Bernoulli random walk ($\Delta x = \pm 1$ with probability 1/2) and the Cauchy random walk, $p(x) = 1/\pi(1 + x^2)$, starting at $x = 0$.

   (a) For the Cauchy walk, show that the displacements can be generated as $x = \tan y$, where $y \in [-\pi/2, \pi/2]$ is uniformly distributed. Plot a few time series of the Cauchy walk, and compare to time series of the Bernoulli walk.

   (b) For each random walk, compute the limiting ($N \to \infty$) PDF of $\alpha_N \in [0, 1]$, the fraction of time steps $1 \le n \le N$ where the walker is in the region $x > 0$. Clearly, $\langle \alpha_N \rangle = \frac{1}{2}$, but this turns out to be the least likely outcome. Which walk has a greater probability of being "ergodic", $\alpha_N \approx 1/2$?

   (c) *Extra credit:* Guess or derive a formula for each limiting distribution.