18.366 Random Walks and Diffusion, Spring 2005, M. Z. Bazant.

Problem Set 3

Due at lecture on Thursday, March 31, 2005.

1. **Modified Kramers-Moyall Expansion.** Let $P_N(x)$ be the probability density for a random walker (or, equivalently, the concentration of a large number of independent walkers) to be at position x at time $t_N = N\tau$. The walker's displacements $(x,t) \to (x', t+\tau)$ are independently chosen with a transition probability $p(x', t+\tau|x,t)$ at regular intervals of time τ. Suppose that the moments, which depend on time and space,

$$M_n(x,t,\tau) = \int p(x+y, t+\tau|x,t) y^n dy \tag{1}$$

are finite. Consider a continuous-time probability density (or concentration), $\rho(x,t)$, satisfying $\rho(x, N\tau) = P_N(x)$.

In class, we formally derived a PDE expansion for $\rho(x,t)$:

$$\frac{\partial \rho}{\partial t} + \sum_{n=2}^{\infty} \frac{\tau^{n-1}}{n!} \frac{\partial^n \rho}{\partial t^n} = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n [D_n(x,t,\tau)\rho(x,t)] \tag{2}$$

where $D_n(x,t,\tau) = M_n(x,t,\tau)/(n!\,\tau)$. Without the second term on the left-hand side, this is called the *Kramers-Moyall expansion*.

In the limit $\tau \to 0$, assume that the transition moments are finite and scale like, $M_1 \sim D_1\tau$, $M_2 \sim 2D_2\tau$, and $M_n \sim M_2^{n/2} = O(\tau^{n/2})$ for $n > 2$ (which follows if the CLT holds on very small time scales). At leading order, we have the *Fokker-Planck equation*,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(D_1\rho) = \frac{\partial^2}{\partial x^2}(D_2\rho) + O(\tau^{1/2}) \tag{3}$$

but please calculate all terms up to $O(\tau)$ in a "modified" expansion of the form

$$\frac{\partial \rho}{\partial t} = \sum_{n=0}^{\infty} \tau^{n/2} L_n \rho \tag{4}$$

where the operators L_n involve only *spatial* derivatives.

2. **Black-Scholes Formulae for Options Prices.**

   (a) Solve the Black-Scholes equation,

$$\frac{\partial w}{\partial t} + rx\frac{\partial w}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 w}{\partial x^2} = rw \tag{5}$$

   backward from maturity, $t < T$, for the long position of a call option, with payoff, $w(x,T) = y(x) = \max(x-K, 0)$.

   (b) Show that the solution is equivalent to a "risk neutral valuation",

$$w(x,t) = e^{-r(T-t)}\langle y(x)\rangle \tag{6}$$

where the expectation is taken with the final value of the underlying asset, $x_T = \int_t^T dx_t$, given by a geometric Brownian motion, which solves the SDE,

$$dx = rxdt + \sigma x dz \tag{7}$$

with volatility σ and mean return, r, the risk free rate (not the actual expected return). [Note: x_T has a lognormal distribution.]

(c) Let $w_c(x,t)$ and $w_p(x,t)$ be the prices of (long) call and put options, respectively, on the same underlying asset, with the same maturity, T, volatility σ, risk-free rate r, and strike price, K. Explain why $w_p(x,t)$ can be found from your solution above, using "put-call parity":

$$w_p(x,t) = w_c(x,t) - x + Ke^{-r(T-t)} \tag{8}$$

3. **Continuum Limit of Bouchaud-Sornette Options Theory.** Consider a discrete random walk for an underlying asset with (additive) independent steps. Assume the displacements $y = \delta x$ in each time step τ have low order moments which depend on the current price,

$$\begin{aligned} \langle \delta x \rangle &= \mu x \delta t & (9)\\ \langle \delta x^2 \rangle &= \sigma^2 x^2 \delta t + \mu^2 x^2 \delta t^2 & (10)\\ \langle \delta x^3 \rangle &= \sigma^3 \lambda_3 x^3 \delta t^{3/2} + 3\mu\sigma^2 x^3 \delta t^2 + O(\delta t^3) & (11)\\ \langle \delta x^4 \rangle &= \sigma^4 (\lambda_4 + 3) x^4 \delta t^2 + O(\delta t^{5/2}). & (12) \end{aligned}$$

As discussed in class this is a general model for random returns in each time step δt.

Assume the Bouchaud-Sornette strategy of minimizing the "quadratic risk" or variance of the return of a position consition of the option and short ϕ of the underlying. Since minimizing the total variance is equivalent to minimizing the variance in each time step, we get the least-squares fit equations given in class, as a recursion for $w(x,t)$.

$$u(x,t) = w(x,t) - \phi(x,t)x = e^{-r\delta t}\left[\int w(x+\delta x, t+\delta t) p(\delta x, \delta t) d\delta x - \phi(x,t) \int (x+\delta x) p(\delta x, \delta t) d\delta x\right] \tag{13}$$

and

$$\phi(x,t) = \frac{1}{\sigma^2 x^2 \delta t} \int (\delta x - \mu x \delta t) w(x+\delta x, t+\delta t) p(\delta x, \delta t) d\delta x \tag{14}$$

Consider the limit $\delta t \to 0$ in these equations and (formally) derive a PDE for $w(x,t)$ and an expression for $\phi(x,t)$ accurate to $O(\delta t^{1/2})$. Each should involve only x derivatives of w (aside from $\partial w/\partial t$ in the PDE). Recover the Black-Scholes equation at leading order $O(1)$. [Extra credit: derive the expansions to $O(\delta t)$. Extra extra credit: Try solving them for a call option!]