

# Lecture 10: Applications in Finance

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After nine lectures concerning the basic mathematics of random walks and the limit of “normal diffusion”, the next three lectures will deal with some applications of what we have learned thus far in the field of financial modeling. In this lecture, we address the following topics:

1. Description of financial time series, including a comparison of normal and lognormal distributions
2. Derivatives, including discussion of:
  - (a) Types of derivatives
  - (b) Derivatives pricing strategies

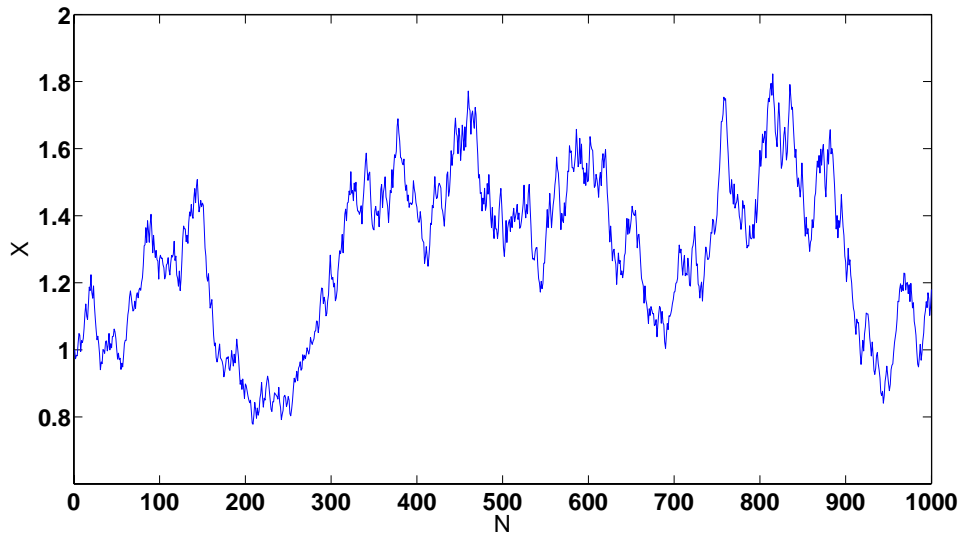
## 1 Financial Time Series

We start this section by defining a financial time series as the value of some financial quantity  $x$  tracked over time. An example of such a time series is shown in Figure 1, which was generated assuming lognormal distribution of steps in the random walk (see below for discussion of this process).  $x$  could represent a stock price or interest rate, for example.

Perhaps the first person to model the financial time series as a random walk was Bachelier (1900) in his PhD thesis. Bachelier describes a random walk,  $X_N$  with IID steps where the time  $t = N\tau$ . The basic time interval,  $\tau$ , could be thought of as the typical time for an independent displacement, since for most markets permit autocorrelations, which decay quickly at the scale  $\tau$ , e.g.

$$\langle X(t)X(t+t') \rangle \sim e^{-\frac{t'}{\tau}} \quad (1)$$

We will consider auto-correlated or “persistent” random walks later in the class. Here, we simply note that  $\tau$  is set by some combination of transaction costs and the ease of transactions (“liquidity”) in a given market. In very liquid markets, like the US Stock Exchange, autocorrelations in stock prices are significant for tens of minutes, with positive correlations below 5 minutes and small negative correlations around 10 minutes, and decaying completely by 30 minutes. Therefore,  $\tau \approx 30$  min is a reasonable time step for the random walk in such a market. (For more analysis of real market data, see the recommended book by Bouchaud and Potters.)



**Figure 1:** Example of possible financial time series.

## 1.1 Additive Random Displacements

Bachelier described the evolution of financial time series as an additive stochastic process, i.e. a simple random walk with  $X_{N+1} = X_N + x_n$ , determined by a sum of IID random displacements. As previous lectures have demonstrated, such a discrete additive process is governed by the Central Limit Theorem (in the central region) as  $N \rightarrow \infty$ , as long as the variance is finite. In the limit of infinitesimal steps, where the CLT is always achieved for any finite time interval, the additive stochastic process is called “Brownian motion” or a “normal process” by mathematicians.

For financial time series, it seems there are two problems with the model of an additive process:

1. Most quantities we wish to model are constrained to have a value greater than zero (e.g. stock prices or interest rates). The additive stochastic process has a finite probability of dropping below zero, thus requiring the introduction of boundary conditions. In practice, however, most financial time series cannot reach zero for obvious reasons.
2. The size of the fluctuations is usually related to the magnitude of the value of the asset, while the additive process assumes a fluctuation size independent of the magnitude of the asset value. Therefore, the random displacements are the same size when the value is very large, as when it is very small. Intuition (and market observation) suggests that fluctuations are roughly proportional to the current value.

For a single asset over a short period of time, these problems are not severe, and the additive process is often an excellent approximation. Over longer periods, however, where the value makes large changes, a different model is needed.

## 1.2 Multiplicative Random Displacements

The problems motivate modeling financial time series as a multiplicative stochastic process, where  $X_{N+1} = X_N y_N$  so that  $X_N = (\prod_{n=1}^N y_n) X_0$ . Consistent with the notion of a continuously compounded interest rate, we might express the multiplicative increment as

$$y_n = e^{r_n \tau} \quad (2)$$

where  $r_n$  is an IID variable describing the rate of return over time  $\tau$ . For small rates of return, the equation for  $y_n$  can be linearized as

$$y_n = (1 + \tilde{r}_n \tau) \quad (3)$$

which is analogous to simple interest (not compounded). By recursion, for the multiplicative increments, Eq. (2), the position of the random walk may be described as

$$X_N = X_0 e^{\sum_{n=1}^N r_n \tau} \quad (4)$$

or

$$\log \frac{X_N}{X_0} = \sum_{n=1}^N r_n \tau \quad (5)$$

Therefore, the logarithm of the value undergoes a simple additive random walk with IID steps.

We now wish to describe the form of the limiting distribution as  $N \rightarrow \infty$ . First, we define  $\mu$ , the *expected rate of return*,  $\mu$ , by  $\langle r_n \tau \rangle = \mu \tau$  and  $\sigma$ , the *volatility*,  $\sigma$ , by  $\text{Var}(r_n \tau) = \sigma^2 \tau$ . Note that  $\mu$  has units of inverse time, while  $\sigma$  has units of inverse square root of time. These definitions are motivated by the additivity of cumulants in Eq. (5); in particular, we expect the mean and variance of the return to grow linearly with time (number of steps).

As  $N \rightarrow \infty$ , the CLT implies that the scaled random variable,  $Z$ , defined as

$$Z_N = \frac{\log \frac{X_N}{X_0} - N \mu \tau}{\sigma \sqrt{N \tau}} \quad (6)$$

converges to a standard normal random variable with PDF,

$$\phi_N(Z) = \phi(Z) = \frac{e^{-Z^2/2}}{\sqrt{2\pi}} \quad (7)$$

in the central region,  $Z = O(1)$ . To convert from  $Z$  back to  $X$  and  $t$ , we account for rescaling,

$$dZ = \frac{dX}{X \sigma \sqrt{N \tau}} \quad (8)$$

and define  $t = N \tau$ . The limiting PDF of  $X(t)$  is the given by a “lognormal density”,

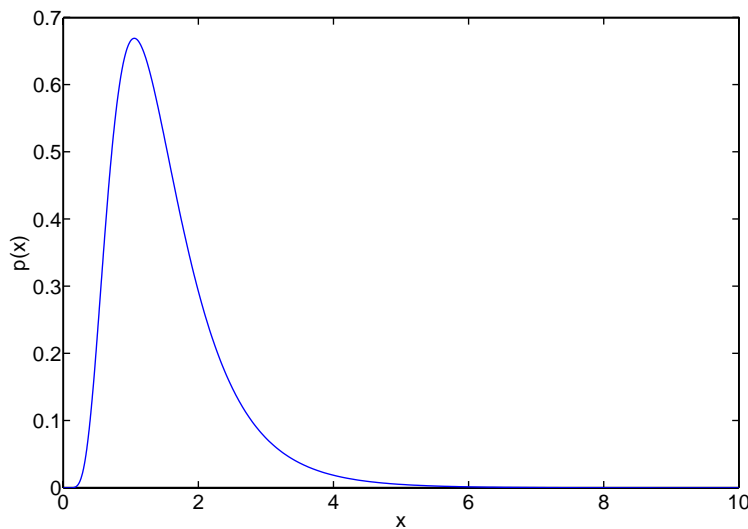
$$\rho(X, t) \sim \frac{\exp\left(-\frac{\log^2(X/\bar{X}_0(t))}{2\sigma^2 t}\right)}{X \sigma \sqrt{2\pi t}} \quad (9)$$

where  $\log \tilde{X}_0(t) = \log X_0 + \mu t$ . In the limit of infinitesimal displacements, where the CLT implies a lognormal distribution for any finite time interval, the multiplicative stochastic process is sometimes called “geometric Brownian motion” or a “lognormal process” by mathematicians.

*Note:* A curious feature of the lognormal density is that it is not uniquely defined by its moments. Feller (recommended reading) points out that the PDF

$$p(x) = \frac{1}{x\sqrt{2\pi}} e^{-\log^2 x/2} (1 + a \sin(2\pi \log(x))) \quad (10)$$

where  $|a| \leq 1$  has moments which do not depend on  $a$ , so it has the same moments as the standard lognormal PDF ( $a = 0$ ). Therefore, in general, *knowing all moments does not uniquely determine a PDF*. Of course, we should expect this from lecture 2, because knowing all the moments implies a complete knowledge of the Taylor series of the (analytic) characteristic function,  $\hat{p}(k)$ , but the series may have a finite radius of convergence, leaving  $\hat{p}(k)$  unconstrained farther from the origin.



**Figure 2:** Lognormal distribution with  $X_0 = 1.0$   $\mu t = 0.3$  and  $\sigma\sqrt{t} = 0.5$

Figure 2 shows the shape of the lognormal PDF. Note that the distribution is generally skewed to favor large values more than small values, which may not be desirable (or accurate) for some quantities like interest rates, but may be reasonable for others, like stock prices, at least over long times. It turns out that real stock prices are better described by a smooth crossover from a normal (additive) random walk for small times to a lognormal (multiplicative) random walk for long times. For the typical liquid stock markets, the crossover occurs at the scale of one month.

The moments are given by

$$m_n = \langle X(t)^n \rangle = \tilde{X}_0(t)^n e^{n^2 \sigma^2 t/2} = X_0^n e^{n\mu t + n^2 \sigma^2 t/2} \quad (11)$$

In particular, the mean drift is

$$m_1 = X_0(t)e^{\mu t + \sigma^2 t/2} \quad (12)$$

Note that even with zero expected return,  $\mu = 0$ , the first moment is greater than zero, and there is a general upward drift. This is a result of nonlinearity in the transformation of the additive return by taking an exponential in Eq. (5), which causes upward random fluctuations to be larger than downward fluctuations, resulting in a net upward bias or drift. Since the volatility  $\sigma$  is a measure of “noise” (randomness in the return), this kind of bias is sometimes called a “noise-induced drift”. (See the recommended book by Risken.) Such upward drift can be inappropriate in many cases, such as interest rates.

## 2 Derivative Securities

### 2.1 Definitions

A derivative is defined as a “financial instruments which derives it’s value from some underlying asset or quantity.” In essence, the value of the derivative is slaved to some other quantity. Derivatives exist for two primary reasons: 1) for speculation and 2) to “hedge” risk.

There are many types of derivatives. Some of the most common are:

1. Forward contract (obligated contract) on an asset  $X(t)$ . A forward contract is an agreement to buy asset  $X$  for a price  $K$  at a future time  $T$ . For the buyer of a forward contract (also known as the long position), the payoff of the contract  $y(X(T))$  is simply the difference between the price of the asset at time  $T$  and the agreed purchase price, i.e.  $y_{LONG}(X(T)) = X(T) - K$ . For the seller (a.k.a. the short position), the payoff is simply the reverse, i.e.  $y_{SHORT}(X(T)) = K - X(T)$ . *Note:* A futures contract is a traded forward contract. *Note:* For more information, see Hall: *Futures, Options, and Derivatives Securities*.
2. Options. A call option represents the *right* (rather than the obligation) to buy an asset  $X(t)$  for a price  $K$  at time  $T$ . As such, the downside risk to the buyer is limited, since the option holder may opt out of purchase if the price  $X(T)$  is less than  $K$ . The long position payoff is  $y_{LONG}(X(T)) = \max(X - K, 0)$ , and the short position in  $y_{SHORT}(X(T)) = \min(X(T) - K, 0)$ .

A put option is the right to sell an asset  $X(T)$  for  $K$ .

3. Foreign exchange rate derivatives: The right to exchange two currencies at a predetermined rate at a future time  $T$ .
4. Interest Rate derivatives: Three common interest rate derivatives are
  - (a) “Caps”: creates a ceiling on floating rate interest costs. When market rates move above the cap rate, the seller pays the purchaser the difference.

- (b) “Swaps”: an agreement to exchange interest payments in a single currency for a stated time period. Only interest payments are exchanged, not principal.
- (c) “Swaptions”: give the holder the right, but not the obligation, to enter into or cancel a swap agreement at a future date.

*Note:* Interest rate derivative definitions taken from

<http://corporate.bmo.com/rm/intrderiv/interestrates/default.asp>

We now turn to the rational theory of pricing derivatives, which is governed by more easily determined principles than the financial time series for the underlying asset. The reason is that even if we do not know the dynamics of underlying asset very well, we can still determine a reasonable price for the derivative, based on correlations between the derivative and the underlying asset (which are known and given by the payoff function at maturity).

## 2.2 Bachelier’s Fair Price

Bachelier first proposed that the price,  $w$ , of an option should be equal to its expected payoff, so that the traders are engaging in a “fair game”. By definition, the price of the option at time  $T$  is given by  $w(T) = y(X(T))$ , so the Bachelier price at time  $t = 0$  is given by:

$$w(0) = w_0 = \langle w \rangle = \langle y(X) \rangle = e^{-rT} \langle y(X(T)) \rangle \quad (13)$$

Here, we add to Bachelier’s original treatment by including the effect of a nonzero interest rate,  $r$ , which should be used to discount a future payoff to the present. This takes into account the “time value of money”, where  $r$  should be some typical interest rate for a very safe (ideally “risk free”) investment. The equation above is equivalent to saying that the value of the derivative at time  $t = 0$  is simply the present value of the future expected value of the option at time  $T$ .

Although it sounds very reasonable, Bachelier’s price is not really “fair”, because it neglects correlations between the derivative and the underlying, which could be exploited by a clever trader. This is most easily illustrated by the case of a forward contract, with a linear payoff function. Consider a short position of a forward contract,  $w$ , with the payoff described by:

$$y(X) = K - X \quad (14)$$

and consider the possibility that a trader buys the underlying and shorts the forward. The total value of the combined portfolio at time  $T$ ,  $u(T)$ , is then given by

$$u(T) = w(T) + X(T) = K - X(T) + X(T) = K \quad (15)$$

Thus, the value of the total portfolio is independent of fluctuations in  $X(T)$ . (We say that the trader has constructed a perfect “hedge” to eliminate risk in the forward contract. Generally, auxiliary trading to reduce risk is referred to as “hedging”.) Taking into account the time value of money, the value of the portfolio at time  $t = 0$  must also be  $Ke^{-rT}$ . Thus

$$u(0) = w_0 + X_0 = Ke^{-rT} \Rightarrow w_0 = Ke^{-rT} - x_0 \quad (16)$$

In contrast, if we neglect the possibility of hedging, then the Bachelier price is

$$w_0 = e^{-rT} \langle y(X(T)) \rangle = e^{-rT} \langle K - X(T) \rangle = e^{-rT} (K - \langle X(T) \rangle) \quad (17)$$

There, the forward is only fairly priced in Bachelier's valuation method in the case of zero expected "excess return" (over the risk-free rate):

$$X_0 = e^{-rT} \langle X(T) \rangle \quad (18)$$

In the next lecture, we will show how to account for the possibility of hedging in a rational theory of pricing derivatives.