Lecture 22: Lévy Distributions

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1 Central Limit Theorem (from previously)

Let $\{x_i\}$ be IID random variables with finite mean μ and <u>finite variance σ^2</u>. Then the RV

$$S_n = \frac{1}{\sigma\sqrt{n}}\left(\sum_{j=1}^{n} x_j - n\mu\right) \tag{1}$$

converges to a standard $N(0,1)$ Gaussian density.

2 Introduction

This lectures covers similar material to Lectures 12 and 13 in the 2003 lecture notes. Also, see Hughes[1] §4.3.

Motivating question: What are the limiting distributions for a sum of IID variables? (And, in particular, those who violate the assumptions of the CLT.)

3 Lévy stability laws

Definitions: $\{x_i\}$ RVs and $X_N = \sum_{n=1}^{N} x_n$

Characteristic function:

$$\hat{p}_n(k) = [\hat{p}(k)]^n \tag{2}$$
$$\hat{p}_{n\times m}(k) = [\hat{p}_m(k)]^n = [\hat{p}(k)]^{n\times m} \tag{3}$$

Introduce new variables $Z_N = \frac{X_N}{a_N}$. The PDF of the Z_N's is given by $F_N(x) = p_N \frac{x}{a_N} \frac{1}{a_N}$.

$$\hat{F}_N(a_N k) = \hat{p}_N(k) \tag{4}$$

Substituting in,

$$\hat{F}_{n\times m}(a_{nm}k) = \left[\hat{F}_m(a_m k)\right]^n \tag{5}$$

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[1] B. Hughes, *Random Walks and Random Environments*, Vol. I, Sec. 2.1 (Oxford, 1995).

As $n \to \infty$, we want $\hat{F}_n(k)$ to tend to some fixed distribution $\hat{F}(k)$. Say that $\lim\limits_{m \to \infty} \frac{a_{nm}}{a_m} = c_n$,

$$\hat{F}(kc_n) = \left[\hat{F}(k)\right]^n \tag{6}$$

Suppose I want to find

$$\hat{F}(k\mu(\lambda)) = \left[\hat{F}(k)\right]^\lambda \tag{7}$$

Introduce $\psi(k) = \log \hat{F}(k)$

$$\psi(k\mu(\lambda)) = \lambda\psi(k) \tag{8}$$

Differentiate:

$$k\mu'(\lambda)\psi'(k\mu(\lambda)) = \psi(\lambda) \tag{9}$$
$$\mu(\lambda = 1) = 1 \tag{10}$$
$$k\mu'(1)\psi'(k) = \psi(k) \tag{11}$$
$$\frac{d\psi}{dk} = \frac{\psi}{\mu'(1)k} \tag{12}$$

This has a solution:

$$\psi(k) = \begin{cases} v_1 |k|^\alpha & k > 0 \\ v_2 |k|^\alpha & k < 0 \end{cases} \tag{13}$$

$$\hat{F}(k) = \begin{cases} \exp(v_1 |k|^\alpha) & k > 0 \\ \exp(v_2 |k|^\alpha) & k < 0 \end{cases} \tag{14}$$

$$v_1, v_2 \in \mathcal{C} \tag{15}$$

Thus we have this condition:

$$\hat{F}(-k) = \hat{F}(k)^* \tag{16}$$
$$\implies \hat{F}(k) = \exp\left((c_1 + ic_2 sgn(k))|k|^\alpha\right) \tag{17}$$
$$c_1, c_2 \in \Re \tag{18}$$

We can also write:

$$\hat{F}(k) = \exp\left[-a |k|^\alpha \left(1 - i\beta \tan\frac{\alpha\pi}{2} sgn(k)\right)\right] \tag{19}$$

In this equation, β relates to the skewness of the distribution.

4 Lévy distribution

For above $\hat{F}$ when $\beta = 0$ (ie. symmetric limiting distribution), you get the Lévy distribution. It is defined in terms of its Fourier transform.

$$\hat{L}_\alpha(a, k) = \hat{F}(k) = \exp(-a |k|^\alpha) \tag{20}$$

Two special cases of the the Lévy distribution are $\alpha = 1, 2$.

$$\alpha = 2 \quad \hat{F}(k) = \exp(-ak^2) \quad \text{like Gaussian but more general} \tag{21}$$
$$\alpha = 1 \quad \hat{F}(k) = \exp(-a |k|) \quad \text{Cauchy} \tag{22}$$

4.1   Large x expansions of the Lévy distribution

$$L_\alpha(a, x) \quad = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \exp ikx - a\,|k|^\alpha \mathrm{d}k \tag{23}$$

$$= \frac{1}{\pi} \int\limits_{0}^{\infty} \cos kx \exp -ak^\alpha \mathrm{d}k \tag{24}$$

Integrate by parts: (25)

$$= \frac{a\alpha}{\pi} \int\limits_{0}^{\infty} \sin \frac{k|x|}{|x|} \exp -ak^\alpha k^{\alpha-1}\mathrm{d}k \quad = \frac{a\alpha}{\pi\,|x|^{1+\alpha}} \int\limits_{0}^{\infty} \xi^{\alpha-1} \sin \xi \exp \frac{-a\xi^\alpha}{|x|^\alpha}\mathrm{d}\xi \tag{26}$$

Where we have made the following substitutions:

$$\xi = k\,|x|\,, \mathrm{d}\xi = \mathrm{d}k\,|x| \tag{27}$$

Taking the limit $x \to \infty$:

$$\sim_{x\to\infty} \quad \frac{a\alpha}{\pi\,|x|^{1+\alpha}} \int\limits_{0}^{\infty} \xi^{\alpha-1} \sin \xi \mathrm{d}\xi \tag{28}$$

$$\sim \quad \frac{a\alpha\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}{\pi\,|x|^{1+\alpha}} \quad\quad \text{power-law tail} \tag{29}$$

4.2   Small x expansion

$$L_\alpha(a, x) \quad\quad = \frac{1}{\pi} \int\limits_{0}^{\infty} \cos kx \exp -ak^\alpha \mathrm{d}k \tag{30}$$

$$\text{Taylor expand cosine:} \quad = \frac{1}{\pi} \int\limits_{0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(kx)^{2n}}{(2n)!} \exp -ak^\alpha \mathrm{d}k \tag{31}$$

$$\text{Substitute:} \quad w = ak^\alpha \quad dw = \alpha ak^{\alpha-1}\mathrm{d}k \tag{32}$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \int\limits_{0}^{\infty} (-1)^n \frac{(x)^{2n}}{(2n)!} \frac{w}{a}^{2n/\alpha} \exp -w \frac{\mathrm{d}w}{\alpha a \frac{w}{a}^{\frac{\alpha-1}{\alpha}}} \tag{33}$$

We're only interested in evaluating this equation as $x \to 0$:

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(x)^{2n}}{a^{\frac{2n+1}{\alpha}}} \Gamma\left(\frac{2n+1}{\alpha}\right) \tag{34}$$

$$\sim \frac{1}{\pi} \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha a^{a/\alpha}} \quad \text{increases as } \alpha \to 0 \tag{35}$$

5   Analogy with Central Limit Theorem?

5.1   Gnedenko-Doblin Theorem

$\frac{1}{a_N} p_N \frac{x}{a_N} \to L_{\alpha,\beta}(a, x)$ *iff* the CDF $P(x)$ satisfies

1. $\lim\limits_{x\to\infty} \frac{p(-x)}{a-p(x)} = \frac{1-\beta}{1+\beta}$

    This is akin to the "amount" of probability on one side of the tail.

2. $\lim\limits_{x \to \infty} \frac{1-p(x)+p(-x)}{1-p(rx)+p(-rx)} = r^\alpha \quad \forall r$

   We can think of this as saying how fast it is decaying.

5.2 Theorem

The distribution $p(x)$ is in the basin of attraction of $L_{\alpha,\beta}(a,x)$ if

$$p(x) \quad \sim \quad \frac{A_\pm}{|x|^{1+\alpha}} \quad x \to \pm\infty \tag{36}$$

$$\beta \quad = \quad \frac{A_+ - A_-}{A_+ + A_-} \tag{37}$$

If $p(x)$ has power-law tails, then the sum of RVs goes to a Lévy distribution.

5.3 Scaling and superdiffusivity

Suppose that $\{x_i\}$ follow a $L_\alpha(a,x)$ distribution, then $\hat{p}(k) = \exp -a\,|k|^\alpha$ and then the characteristic function at $x_N = \sum\limits_{n=1}^{N} x_n$ is given by

$$\hat{p}_N(k) \quad = [\exp -a\,|k|^\alpha]^N \quad = \exp -a\,|k|^\alpha\,N \tag{38}$$

$$p_N(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \exp ikx - aN\,|k|^\alpha dk \tag{39}$$

Now we substitute:

$$k = VN^{-1/\alpha}, \mathrm{d}k = -\mathrm{d}VN^{-1/\alpha} \tag{40}$$

$$p_N(k) \quad = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \exp iV\left(\frac{x}{N^{1/\alpha}}\right) - a\,|V|^\alpha \frac{\mathrm{d}V}{N^{1/\alpha}} \quad = \frac{1}{N^{1/\alpha}} L_\alpha\left(a, \frac{x}{N^{1/\alpha}}\right) \tag{41}$$

Thus, the width scales like $N^{1/\alpha}$.

   If $\alpha > 2$, we get scaling that exceeds "square-root" scaling, ie. a superdiffusive process.