MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Computation

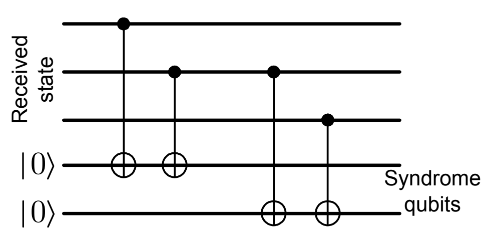
Problem 1. Single-qubit σ_X errors can project the codeword $\alpha|000\rangle + \beta|111\rangle$ onto one of the following subspaces: $\{|000\rangle,|111\rangle\}$, $\{|100\rangle,|011\rangle\}$, $\{|010\rangle,|101\rangle\}$, and $\{|001\rangle,|110\rangle\}$. Construct a quantum circuit that specifies in which subspace the received codeword is. You can use two work qubits, some operations on the work qubits and the original qubits, and finally, a measurement on the work space.

Solution:

A repetition code is like a linear block code with the generator matrix $G = \begin{bmatrix}1 & 1 & 1\end{bmatrix}$. Therefore the parity check matrix consists of two row vectors orthogonal to (1 1 1). For instance, let's choose

$$H = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}.$$

Then, for a received vector $\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}$, the syndromes are $x_1 + x_2$ and $x_2 + x_3$ (mod 2). The same idea can be used for the quantum code where we use CNOTs to implement XOR operation. It can be easily seen that the following circuit



provides us with the right syndrome. For instance, assuming occurrence of $\sigma_X^{(1)}$ error, the received state is $\alpha|100\rangle + \beta|011\rangle$. For this state, the output of the above circuit is $(\alpha|100\rangle + \beta|011\rangle) \otimes |10\rangle$. So, by measuring the work qubits, this error can be detected and corrected by applying $\sigma_X^{(1)}$ to the received state. For $\sigma_X^{(2)}$, the syndrome is $|11\rangle$, and for $\sigma_X^{(3)}$, it is $|01\rangle$. For the no-error case, the syndrome is $|00\rangle$.

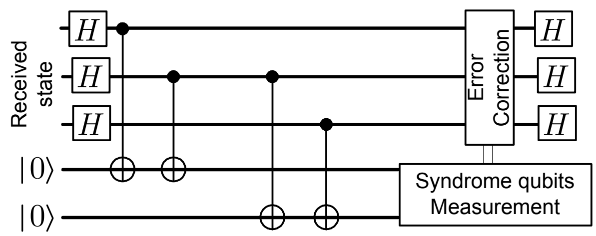
Problem 2. Show how to correct a single σ_Z error for the phase-error correcting code:

$$|0\rangle \rightarrow \frac{1}{2}(|000\rangle + |110\rangle + |101\rangle + |011\rangle)$$

$$|1\rangle \to \frac{1}{2}(|111\rangle + |001\rangle + |010\rangle + |100\rangle).$$

Solution:

Remembering that $|0\rangle \to (|+++\rangle + |---\rangle)/2$ and $|1\rangle \to (|+++\rangle - |---\rangle)/2$ we just need to use Hadamard gates to take the received state from $|\pm\rangle$-space to $|0/1\rangle$-space, use the circuit in Problem 1 to detect and correct the error, and use Hadamard gates again to get the corrected received state back.



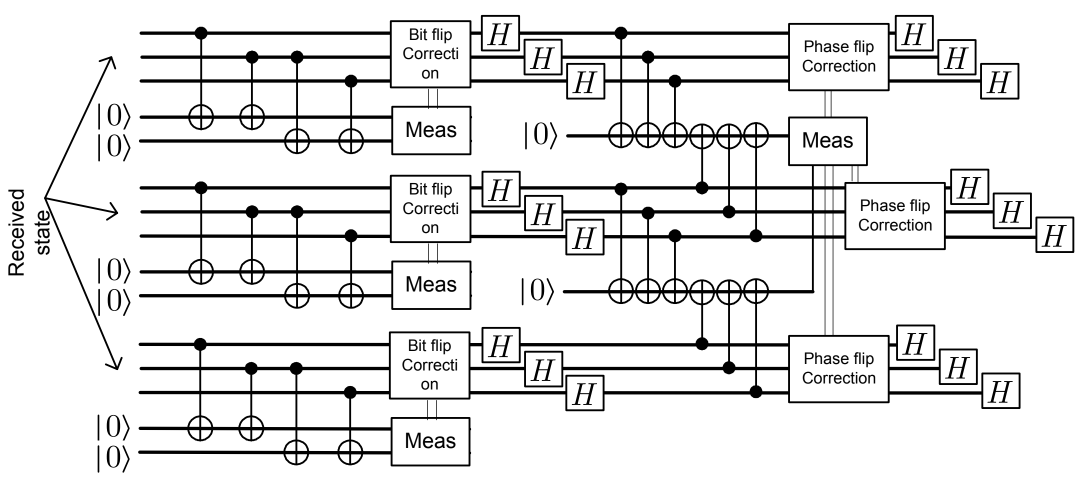
Problem 3. For the Shor's nine-qubit code:

$$\begin{aligned}|0\rangle \to \frac{1}{2}(&|000\rangle|000\rangle|000\rangle + |000\rangle|111\rangle|111\rangle \\ &+|111\rangle|000\rangle|111\rangle + |111\rangle|111\rangle|000\rangle) \\ |1\rangle \to \frac{1}{2}(&|000\rangle|000\rangle|111\rangle + |000\rangle|111\rangle|000\rangle \\ &+|111\rangle|000\rangle|000\rangle + |111\rangle|111\rangle|111\rangle),\end{aligned}$$

give a quantum circuit that corrects a possible single-qubit Pauli error.

Solution:

It's a cascade of bit-flip error correction and phase-flip error correction:



Problem 4. For the quantum Hamming code, show that the vector $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ gets encoded to

$$|\psi_{-}\rangle = \frac{1}{4}\left(\sum_{x\in\{H\}} |x\rangle - \sum_{x\in\{G\}-\{H\}} |x\rangle \right),$$

where $\{H\}$ is the corresponding binary subspace spanned by

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

and $\{G\}-\{H\}$ is the set of complements of $\{H\}$, which is all elements of $\{H\}$+[1 1 1 1 1 1 1]. Then, show that

$$H^{\otimes 7}|\psi_{-}\rangle = \frac{1}{\sqrt{8}} \sum_{x\in\{G\}-\{H\}} |x\rangle .$$

Solution:

For the quantum Hamming code, we have

$$|0\rangle \to |\psi_0\rangle = \frac{1}{\sqrt{8}} \sum_{x\in\{H\}} |x\rangle$$

$$|1\rangle \to |\psi_1\rangle = \frac{1}{\sqrt{8}} \sum_{x\in\{G\}-\{H\}} |x\rangle$$

Therefore,

$$|-\rangle \to (|\psi_0\rangle - |\psi_1\rangle)/\sqrt{2} .$$

Now,

$$\begin{aligned}
H^{\otimes 7}|\psi_{-}\rangle &= \frac{1}{\sqrt{2^{10}}}\left(\sum_{x\in\{H\}} \sum_{y} (-1)^{x\cdot y}|y\rangle - \sum_{x\in\{G\}-\{H\}} \sum_{y} (-1)^{x\cdot y}|y\rangle \right) \\
&= \frac{1}{\sqrt{2^{10}}}\left(\sum_{x\in\{H\}} \sum_{y} (-1)^{x\cdot y}|y\rangle - \sum_{x\in\{H\}} \sum_{y} (-1)^{x\cdot y + w(y)}|y\rangle \right), \\
&\qquad\qquad w(y) \equiv \textit{Hamming Weight } (y) \\
&= \frac{1}{\sqrt{2^{10}}}\left(|\{H\}| \sum_{y\in\{G\}} (1-(-1)^{w(y)})|y\rangle \right), \\
&\qquad\qquad \text{using Ex. 10.25 for } C = \{H\} \Rightarrow \{H\}^{\perp} = \{G\} \\
&= \frac{|\{H\}|}{\sqrt{2^{10}}}\left(\sum_{y\in\{H\}} (1-(-1)^{0})|y\rangle + \sum_{y\in\{G\}-\{H\}} (1-(-1)^{1})|y\rangle \right)
\end{aligned}$$

$$= \frac{1}{\sqrt{8}} \sum_{y \in \{G\}-\{H\}} |y\rangle,$$

where we used the fact that all codewords in {H} have even weights and all codewords in {G}-{H} have odd weights.