18.440: Lecture 1

Permutations and combinations, Pascal's triangle, learning to count

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Outline

Remark, just for fun

Permutations

Counting tricks

Binomial coefficients

Problems

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Politics

- Suppose that betting markets place the probability that your favorite presidential candidates will be elected at 58 percent. Price of a contact that pays 100 dollars if your candidate wins is 58 dollars.
- Market seems to say that your candidate will probably win, if "probably" means with probability greater than .5.
- The price of such a contract may fluctuate in time.
- Let $X(t)$ denote the price at time t.
- Suppose $X(t)$ is known to vary continuously in time. What is the probability it will reach 59 before reaching 57?
- "Efficient market hypothesis" suggests about .5.
- Reasonable model: use sequence of fair coin tosses to decide the order in which $X(t)$ passes through different integers.

Which of these statements is "probably" true?

- 1. $X(t)$ will go below 50 at some future point.
- 2. $X(t)$ will get all the way below 20 at some point
- 3. $X(t)$ will reach both 70 and 30, at different future times.
- 4. $X(t)$ will reach both 65 and 35 at different future times.
- 5. $X(t)$ will hit 65, then 50, then 60, then 55.
- Answers: 1, 2, 4.
- Full explanations coming at the end of the course.
- Point for now is that probability is everywhere: politics, military, finance and economics, all kinds of science and engineering, philosophy, religion, making cool new cell phone features work, social networking, dating websites, etc.
- All of the math in this course has a lot of applications.

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Permutations

- How many ways to order 52 cards?
- Answer: $52 \cdot 51 \cdot 50 \cdot \ldots \cdot 1 = 52! = 80658175170943878571660636856403766975289505440883277824 \times 10^{12}$
- n hats, n people, how many ways to assign each person a hat?
- Answer: $n!$
- n hats, $k < n$ people, how many ways to assign each person a hat?
- $n \cdot (n-1) \cdot (n-2) \ldots (n-k+1) = n!/(n-k)!$
- A **permutation** is a map from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. There are $n!$ permutations of n elements.

Permutation notation

- A **permutation** is a function from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ whose range is the whole set $\{1, 2, \ldots, n\}$. If σ is a permutation then for each j between 1 and n, the the value $\sigma(j)$ is the number that j gets mapped to.
- For example, if $n = 3$, then σ could be a function such that $\sigma(1) = 3$, $\sigma(2) = 2$, and $\sigma(3) = 1$.
- If you have n cards with labels 1 through n and you shuffle them, then you can let $\sigma(j)$ denote the label of the card in the jth position. Thus orderings of n cards are in one-to-one correspondence with permutations of n elements.
- One way to represent σ is to list the values $\sigma(1), \sigma(2), \ldots, \sigma(n)$ in order. The σ above is represented as $\{3, 2, 1\}$.
- If σ and ρ are both permutations, write $\sigma \circ \rho$ for their composition. That is, $\sigma \circ \rho(j) = \sigma(\rho(j))$.

Cycle decomposition

- Another way to write a permutation is to describe its cycles:
- For example, taking $n = 7$, we write $(2, 3, 5), (1, 7), (4, 6)$ for the permutation σ such that $\sigma(2) = 3, \sigma(3) = 5, \sigma(5) = 2$ and $\sigma(1) = 7, \sigma(7) = 1$, and $\sigma(4) = 6, \sigma(6) = 4$.
- If you pick some j and repeatedly apply σ to it, it will "cycle through" the numbers in its cycle.
- Generally, a function is called an **involution** if $f(f(x)) = x$ for all x.
- A permutation is an involution if all cycles have length one or two.
- A permutation is "fixed point free" if there are no cycles of length one.

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Fundamental counting trick

- n ways to assign hat for the first person. No matter what choice I make, there will remain $n-1$ was to assign hat to the second person. No matter what choice I make there, there will remain $n-2$ ways to assign a hat to the third person, etc.
- This is a useful trick: break counting problem into a sequence of stages so that one always has the same number of choices to make at each stage. Then the total count becomes a product of number of choices available at each stage.
- Easy to make mistakes. For example, maybe in your problem, the number of choices at one stage actually *does* depend on choices made during earlier stages.

Another trick: overcount by a fixed factor

- If you have 5 indistinguishable black cards, 2 indistinguishable red cards, and three indistinguishable green cards, how many distinct shuffle patterns of the ten cards are there?
- Answer: if the cards were distinguishable, we'd have 10!. But we're overcounting by a factor of 5!2!3!, so the answer is $10!/(5!2!3!)$.

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$\binom{n}{k}$ notation

- How many ways to choose an ordered sequence of k elements from a list of n elements, with repeats allowed?
- Answer: n^k
- How many ways to choose an ordered sequence of k elements from a list of n elements, with repeats forbidden?
- Answer: $n!/(n-k)!$
- How many way to choose (unordered) k elements from a list of n without repeats?
- Answer: $\binom{n}{k} := \frac{n!}{k!(n-k)!}$
- What is the coefficient in front of x^k in the expansion of $(x+1)^n$?
- Answer: $\binom{n}{k}$.

Pascal's triangle

- Arnold principle.
- A simple recursion: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
- What is the coefficient in front of x^k in the expansion of $(x+1)^n$?
- Answer: $\binom{n}{k}$.
- $(x+1)^n = \binom{n}{0} \cdot 1 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$.
- Question: what is $\sum_{k=0}^{n} \binom{n}{k}$?
- Answer: $(1+1)^n = 2^n$.

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More problems

- How many full house hands in poker?
- $13\binom{4}{3} \cdot 12\binom{4}{2}$
- How many "2 pair" hands?
- $13\binom{4}{2} \cdot 12\binom{4}{2} \cdot 11\binom{4}{1}/2$
- How many royal flush hands?
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More problems

- How many hands that have four cards of the same suit, one card of another suit?
- $4\binom{13}{4} \cdot 3\binom{13}{1}$
- How many 10 digit numbers with no consecutive digits that agree?
- If initial digit can be zero, have $10 \cdot 9^9$ ten-digit sequences. If initial digit required to be non-zero, have 9^{10}.
- How many 10 digit numbers (allowing initial digit to be zero) in which only 5 of the 10 possible digits are represented?
- This is one is tricky, can be solved with *inclusion-exclusion* (to come later in the course).
- How many ways to assign a birthday to each of 23 distinct people? What if no birthday can be repeated?
- 366^{23} if repeats allowed. $366!/343!$ if repeats not allowed.

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