18.440: Lecture 32

Strong law of large numbers and Jensen's inequality

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Outline

A story about Pedro

Strong law of large numbers

Jensen's inequality

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Pedro's hopes and dreams

- Pedro is considering two ways to invest his life savings.
- One possibility: put the entire sum in government insured interest-bearing savings account. He considers this completely risk free. The (post-tax) interest rate equals the inflation rate, so the real value of his savings is guaranteed not to change.
- Riskier possibility: put sum in investment where every month *real* value goes up 15 percent with probability .53 and down 15 percent with probability .47 (independently of everything else).
- How much does Pedro make in expectation over 10 years with risky approach? 100 years?

Pedro's hopes and dreams

- How much does Pedro make in expectation over 10 years with risky approach? 100 years?
- Answer: let R_i be i.i.d. random variables each equal to 1.15 with probability .53 and .85 with probability .47. Total value after n steps is initial investment times $T_n := R_1 \times R_2 \times \ldots \times R_n$.
- Compute $E[R_1] = .53 \times 1.15 + .47 \times .85 = 1.009$.
- Then $E[T_{120}] = 1.009^{120} \approx 2.93$. And $E[T_{1200}] = 1.009^{1200} \approx 46808.9$

Pedro's financial planning

- How would you advise Pedro to invest over the next 10 years if Pedro wants to be completely sure that he doesn't lose money?
- What if Pedro is willing to accept substantial risk if it means there is a good chance it will enable his grandchildren to retire in comfort 100 years from now?
- What if Pedro wants the money for himself in ten years?
- Let's do some simulations.

Logarithmic point of view

- We wrote $T_n = R_1 \times \ldots \times R_n$. Taking logs, we can write $X_i = \log R_i$ and $S_n = \log T_n = \sum_{i=1}^n X_i$.
- Now S_n is a sum of i.i.d. random variables.
- $E[X_1] = E[\log R_1] = .53(\log 1.15) + .47(\log .85) \approx -.0023$.
- By the law of large numbers, if we take n extremely large, then $S_n/n \approx -.0023$ with high probability.
- This means that, when n is large, S_n is *usually* a very negative value, which means T_n is *usually* very close to zero (even though its expectation is very large).
- Bad news for Pedro's grandchildren. After 100 years, the portfolio is probably in bad shape. But what if Pedro takes an even longer view? Will T_n converge to zero with probability one as n gets large? Or will T_n perhaps always *eventually* rebound?

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Strong law of large numbers

- Suppose X_i are i.i.d. random variables with mean μ.
- Then the value $A_n := \frac{X_1+X_2+\ldots+X_n}{n}$ is called the *empirical average* of the first n trials.
- Intuition: when n is large, A_n is typically close to μ.
- Recall: **weak law of large numbers** states that for all $\epsilon > 0$ we have $\lim_{n\to\infty} P\{|A_n - \mu| > \epsilon\} = 0$.
- The **strong law of large numbers** states that with probability one $\lim_{n\to\infty} A_n = \mu$.
- It is called "strong" because it implies the weak law of large numbers. But it takes a bit of thought to see why this is the case.

Strong law implies weak law

- Suppose we know that the strong law holds, i.e., with probability 1 we have $\lim_{n\to\infty} A_n = \mu$.
- Strong law implies that for every ϵ the random variable $Y_\epsilon = \max\{n : |A_n - \mu| > \epsilon\}$ is finite with probability one. It has some probability mass function (though we don't know what it is).
- Note that if $|A_n - \mu| > \epsilon$ for some n value then $Y_\epsilon \geq n$.
- Thus for each n we have $P\{|A_n - \mu| > \epsilon\} \leq P\{Y_\epsilon \geq n\}$.
- So $\lim_{n\to\infty} P\{|A_n - \mu| > \epsilon\} \leq \lim_{n\to\infty} P\{Y_\epsilon \geq n\} = 0$.
- If the right limit is zero for each ϵ (strong law) then the left limit is zero for each ϵ (weak law).

Proof of strong law assuming $E[X^4] < \infty$

- Assume $K := E[X^4] < \infty$. Not necessary, but simplifies proof.
- Note: $\mathrm{Var}[X^2] = E[X^4] - E[X^2]^2 > 0$, so $E[X^2]^2 \le K$.
- The strong law holds for i.i.d. copies of X if and only if it holds for i.i.d. copies of $X - \mu$ where μ is a constant.
- So we may as well assume $E[X] = 0$.
- Key to proof is to bound fourth moments of A_n.
- $E[A_n^4] = n^{-4}E[S_n^4] = n^{-4}E[(X_1 + X_2 + \ldots + X_n)^4]$.
- Expand $(X_1 + \ldots + X_n)^4$. Five kinds of terms: $X_iX_jX_kX_l$ and $X_iX_jX_k^2$ and $X_iX_j^3$ and $X_i^2X_j^2$ and X_i^4.
- The first three terms all have expectation zero. There are $\binom{n}{2}$ of the fourth type and n of the last type, each equal to at most K. So $E[A_n^4] \le n^{-4}\Big(6\binom{n}{2} + n\Big)K$.
- Thus $E[\sum_{n=1}^\infty A_n^4] = \sum_{n=1}^\infty E[A_n^4] < \infty$. So $\sum_{n=1}^\infty A_n^4 < \infty$ (and hence $A_n \to 0$) with probability 1.

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Jensen's inequality statement

- Let X be random variable with finite mean $E[X] = \mu$.
- Let g be a **convex** function. This means that if you draw a straight line connecting two points on the graph of g, then the graph of g lies below that line. If g is twice differentiable, then convexity is equivalent to the statement that $g''(x) \geq 0$ for all x. For a concrete example, take $g(x) = x^2$.
- **Jensen's inequality:** $E[g(X)] \geq g(E[X])$.
- Similarly, if g is **concave** (which means $-g$ is convex), then $E[g(X)] \leq g(E[X])$.
- If your utility function is concave, then you always prefer a safe investment over a risky investment with the same expected return.

More about Pedro

- Disappointed by the strong law of large numbers, Pedro seeks a better way to make money.
- Signs up for job as "hedge fund manager". Allows him to manage $C \approx 10^9$ dollars of somebody else's money. At end of each year, he and his staff get two percent of principle plus twenty percent of profit.
- Precisely: if X is end-of-year portfolio value, Pedro gets

$$g(X) = .02C + .2 \max\{X - C, 0\}.$$

- Pedro notices that g is a convex function. He can therefore increase his expected return by adopting risky strategies.
- Pedro has strategy that increases portfolio value 10 percent with probability .9, loses everything with probability .1.
- He repeats this yearly until fund collapses.
- With high probability Pedro is rich by then.

Perspective

- The "two percent of principle plus twenty percent of profit" is common in the hedge fund industry.
- The idea is that fund managers have both guaranteed revenue for expenses (two percent of principle) and incentive to make money (twenty percent of profit).
- Because of Jensen's inequality, the convexity of the payoff function is a genuine concern for hedge fund investors. People worry that it encourages fund managers (like Pedro) to take risks that are bad for the client.
- This is a special case of the "principal-agent" problem of economics. How do you ensure that the people you hire genuinely share your interests?

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