18.443. Practice test 1.

Consider the family of distributions with p.d.f.

$$f(x|\theta) = \theta x^{\theta-1}, \quad \text{for} \quad 0 < x < 1, \quad \text{and } \theta > 0. \tag{1}$$

Consider an i.i.d. sample $X_1, \ldots, X_n$ from this distribution. As always, the underlying parameter θ for this sample is unknown. In problems (1) and (2) the distribution is given by equation (1) above.

(1) Find the MLE $\hat{\theta}$ of θ.

(2) Compute Fisher information $I(\theta)$ and state asymptotic normality of MLE $\hat{\theta}$. If $n = 100$, find c such that

$$\Pr(-c \le \hat{\theta} - \theta \le c) \approx 0.95.$$

(3) Suppose that a covariance matrix

$$\Sigma = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Distribution $N(0, \Sigma)$ has what density in what basis?

(4) Suppose that a sample $X_1, \ldots, X_{15} \sim N(\mu_x, \sigma_x^2)$, where μ_x and σ_x^2 are unknown, has sample mean and sample variance

$$\hat{\mu}_x = \bar{X} = 2.4, \ \ \hat{\sigma}_x^2 = \bar{X^2} - \bar{X}^2 = 0.55.$$

Find 95% confidence intervals for μ and σ^2.

(5) In addition to the sample from problem (4) suppose that we are given a sample $Y_1, \ldots, Y_{10} \sim N(\mu_y, \sigma_x^2)$ from a distribution with the same variance as Xs, i.e. $\sigma_x^2 = \sigma_y^2$, but possibly different mean μ_y. Suppose that

$$\hat{\mu}_y = \bar{Y} = 2.8, \ \ \hat{\sigma}_y^2 = \bar{Y^2} - \bar{Y}^2 = 0.37.$$

Find a 90% confidence interval for $\mu_x - \mu_y$.

(6) For samples in problems (4) and (5), perform the t-test of the hypothesis that $\mu_x \le \mu_y$ under the assumption of equal variances. Test whether the variances are equal using the F-test. In both tests, find a p-value and use level of significance $\alpha = 0.05$.

(7) If $X_1, \ldots, X_5$ are i.i.d. exponential $E(\alpha)$, what is the distribution of $X_1 + \ldots + X_5$. If $X \sim \Gamma(\alpha, \beta)$ what is the distribution of cX where $c > 0$ is a constant.

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