18.445 2015 Appendix: Almost Sure Martingale Convergence Theorem

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Theorem 1. *Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space with a filtration $(\mathscr{F}_n)_{n \geq 0}$. Let $X = (X_n)_{n \geq 0}$ be a supermartingale which is bounded in L^1, i.e. $\sup_n \mathbb{E}[|X_n|] < \infty$. Then*

$$X_n \to X_\infty, \quad a.s. \quad \text{as } n \to \infty$$

where $X_\infty \in L^1(\Omega, \mathscr{F}_\infty, \mathbb{P})$ with $\mathscr{F}_\infty = \sigma(\mathscr{F}_n, n \geq 0)$.

Let $x = (x_n)_{n \geq 0}$ be a sequence of real numbers. Let $a < b$ be two real numbers. We define $T_0(x) = 0$ and inductively, for $k \geq 0$,

$$S_{k+1}(x) = \inf\{n \geq T_k(x) : x_n \leq a\}, \quad T_{k+1}(x) = \inf\{n \geq S_{k+1}(x) : x_n \geq b\},$$

with the usual convention that $\inf \emptyset = \infty$.

Define the number of upcrossings of $[a,b]$ by x by time n to be

$$N_n([a,b],x) = \sup\{k \geq 0 : T_k(x) \leq n\}.$$

As $n \uparrow \infty$, we have

$$N_n([a,b],x) \uparrow N([a,b],x) = \sup\{k \geq 0 : T_k(x) < \infty\},$$

which is the total number of upcrossings of $[a,b]$ by x.

Lemma 2. *A sequence of real numbers x converges in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ if and only if*

$$N([a,b],x) < \infty \quad \text{for all rationals } a < b.$$

Lemma 3. *[Doob's upcrossing inequality] Let X be a supermartingale and $a < b$ be two real numbers. Then, for all $n \geq 0$,*

$$(b-a)\mathbb{E}[N_n([a,b],X)] \leq \mathbb{E}[(a - X_n)^+].$$

Proof. To simplify the notations, we write

$$T_k = T_k(X), \quad S_k = S_k(X), \quad N = N_n([a,b],X).$$

On the one hand, by the definition of (T_k) and (S_k), we have that, for all $k \geq 1$,

$$X_{T_k} - X_{S_k} \geq b - a. \tag{1}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{k=1}^{n}\left(X_{T_k\wedge n}-X_{S_k\wedge n}\right)\\
&\quad= \sum_{k=1}^{N}\left(X_{T_k}-X_{S_k}\right)+\sum_{k=N+1}^{n}\left(X_n-X_{S_k\wedge n}\right)\\
&\quad= \sum_{k=1}^{N}\left(X_{T_k}-X_{S_k}\right)+\left(X_n-X_{S_{N+1}}\right)1_{[S_{N+1}\le n]}. \quad \text{(Note that } T_N\le n, S_{N+1}<T_{N+1}<S_{N+2}).
\end{aligned}$$

Since (T_k) and (S_k) are stopping times, we have that $S_k\wedge n\le T_k\wedge n$ are bounded stopping times. Therefore, by *Optional Stopping Theorem*, we have

$$\mathbb{E}[X_{S_k\wedge n}]\ge\mathbb{E}[X_{T_k\wedge n}],\quad \forall k.$$

Combining with Equation (1), we have

$$0\ge\mathbb{E}\left[\sum_{k=1}^{n}\left(X_{T_k\wedge n}-X_{S_k\wedge n}\right)\right]\ge(b-a)\mathbb{E}[N]-\mathbb{E}[(a-X_n)^+],$$

since $\left(X_n-X_{S_{N+1}}\right)1_{[S_{N+1}\le n]}\ge-(a-X_n)^+$. This implies the conclusion. □

Proof of Theorem 1. Let $a<b$ be rationals. By Lemma 3, we have that

$$\mathbb{E}[N_n([a,b],X)]\le\frac{\mathbb{E}[(a-X_n)^+]}{b-a}\le\frac{\mathbb{E}[|X_n|]+a}{b-a}.$$

By *Monotone Convergence Theorem*, we have that

$$\mathbb{E}[N([a,b],X)]\le\frac{\sup_n\mathbb{E}[|X_n|]+a}{b-a}<\infty.$$

Therefore, we have almost surely that $N([a,b],X)<\infty$. Write

$$\Omega_0=\cap_{a<b\in\mathbb{Q}}[N([a,b],X)<\infty].$$

Then $\mathbb{P}[\Omega_0]=1$. By Lemma 2 on Ω_0, we have that X converges to a possibly infinite limit. Set

$$X_\infty=\begin{cases}\lim_n X_n, & \text{on } \Omega_0,\\ 0 & \text{on } \Omega\setminus\Omega_0.\end{cases}$$

Then X_∞ is $\mathscr{F}_\infty$-measurable and by *Fatou's Lemma*, we have

$$\mathbb{E}[|X_\infty|]\le\mathbb{E}[\liminf_n|X_n|]\le\sup_n\mathbb{E}[|X_n|]<\infty.$$

Therefore $X_\infty\in L^1$. □