

# 18.445 Introduction to Stochastic Processes

## Lecture 19: Galton-Watson tree

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# Rooted trees

A **tree** is a connected graph with no cycles.

A **rooted tree** has a distinguished vertex  $v_0$ , called the root.

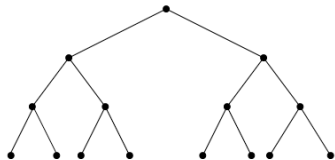
The **depth** of a vertex  $v$  is its graph distance to the root.

A **leaf** is a vertex with degree one.

Consider a **regular rooted tree** :

- each vertex has a fixed number (say  $m$ ) of offspring
- $Z_n$  : the number vertices in the  $n$ -th generation
- for regular tree :  $Z_n = m^n$

A rooted binary tree



In real life, we often encounter trees where the number of offspring of a vertex is random.

# Galton-Watson tree

- It starts with one initial ancestor
- it produces a certain number of offspring according to some distribution  $\mu$
- the new particles form the first generation
- each of the new particles produces offspring according to  $\mu$ , independently of each other
- the system regenerates
- $Z_n$  : the number of particles in  $n$ -th generation

**Observation :** If  $Z_n = 0$  for some  $n$ , then  $Z_m = 0$  for all  $m \geq n$   
→ the family become extinct

**Question :** extinction probability  $q = \mathbb{P}[Z_n = 0 \text{ eventually}]$  ?

# Extinction probability

## Notations :

- $\mu$  : let  $p_k$  be the probability that a particle has  $k$  children,  $k \geq 0$
- $\sum_0^\infty p_k = 1$
- $m := \mathbb{E}[Z_1] = \sum_0^\infty kp_k$
- Assume  $p_0 + p_1 < 1$ .
- Convention  $0^0 = 1$
- extinction probability  $q = \mathbb{P}[Z_n = 0 \text{ eventually}]$
- $f$  : the generating function of the reproduction law :

$$f(s) := \mathbb{E}[s^{Z_1}] = \sum_0^\infty s^k p_k.$$

- $f(0) = p_0, f(1) = 1, f'(1) = m.$

## Theorem

*The extinction probability  $q$  is the smallest root of  $f(s) = s$  for  $s \in [0, 1]$ . In particular,  $q = 1$  if  $m \leq 1$ , and  $q < 1$  if  $m > 1$ .*

# Extinction probability

## Theorem

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- In the subcritical case ( $m < 1$ ),  
the GW tree dies out with probability 1
- In the critical case ( $m = 1$ ),  
the GW tree dies out with probability 1
- In the supercritical case ( $m > 1$ ),  
the GW tree survives with strictly positive probability  $1 - q$ .

**Question :** In the supercritical case  $m > 1$ , how fast the tree grows ?  
We know that  $\mathbb{E}[Z_n] = m^n$ , do we have  $Z_n \sim m^n$  ?

# Growth rate

**Assumption** :  $m \in (1, \infty)$ . Define  $W_n = Z_n/m^n$ .

- $(W_n)_{n \geq 0}$  is a non-negative martingale
- $W_n \rightarrow W$  a.s.
- By Fatou's Lemma, we have  $\mathbb{E}[W] \leq 1$

**Observation** : If  $W > 0$ , then  $Z_n \sim m^n$ ; if  $W = 0$ , then  $Z_n \ll m^n$ .

Theorem (Kesten and Stigum)

$$\mathbb{E}[W] = 1 \Leftrightarrow \mathbb{P}[W > 0 \mid \text{non-extinction}] \Leftrightarrow \mathbb{E}[Z_1 \log^+ Z_1] < \infty$$

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## Theorem

If  $\mathbb{E}[Z_1^2] < \infty$ , then  $\mathbb{E}[W] = 1$  and  $\mathbb{P}[W = 0] = q$ .

## Lemma

$\mathbb{P}[W = 0]$  is either  $q$  or 1.

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