Pset is due on Friday, April 2nd. This time there will be no extension.

(1) Consider a set $\Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_d) : \sum_{i=1}^d |\lambda_i| \leq 1\} \subset R^d$ and a distance $d(\lambda^1, \lambda^2) = \sum_{i\leq d} |\lambda_i^1 - \lambda_i^2|$. Prove that $D(\Lambda, \varepsilon, d) \leq (4/\varepsilon)^d$.

(2) Consider a class of functions $\mathcal{F} = \{f : \mathcal{X} \to [0,1]\}$ such that

$$\forall x = (x_1, \ldots, x_n) \in \mathcal{X}^n \quad \log D(\mathcal{F}, \varepsilon, d_x) \leq KV \log \frac{2}{\varepsilon}.$$

Prove that

$$\mathbb{E} \sup_{f\in\mathcal{F}} \Big(\mathbb{E}f - n^{-1} \sum_{i=1}^n f(x_i)\Big) \leq K\Big(\frac{V}{n}\Big)^{1/2}$$

and

$$\mathbb{E} \sup_{f\in\mathcal{F}} \frac{\mathbb{E}f - n^{-1}\sum_{i=1}^n f(x_i)}{\sqrt{\mathbb{E}f}} \leq K\Big(\frac{V \log n}{n}\Big)^{1/2}.$$

(3) Consider a class of functions $\mathcal{F} = \{f : \mathcal{X} \to [0,1]\}$ such that

$$\forall x = (x_1, \ldots, x_n) \in \mathcal{X}^n \quad \log D(\mathcal{F}, \varepsilon, d_x) \leq \Big(\frac{K}{\varepsilon}\Big)^\alpha, \qquad \text{where } 0 < \alpha < 2.$$

Assume that for some function $f \in \mathcal{F}$ which may depend on the data we have $\sum_{i=1}^n f(x_i) = 0$. Prove that for any $t > 0$ with probability at least $1 - e^{-t}$

$$\mathbb{E}f \leq K\Big(n^{-2/(2+\alpha)} + \frac{t}{n}\Big).$$

(4) Given $x_1, \ldots, x_n \in \mathcal{X}$ and a family of functions $\mathcal{F} = \{f : \mathcal{X} \to \mathbb{R}\}$, consider two distances on $\mathcal{F}$,

$$d_1(f,g) = \frac{1}{n}\sum_{i=1}^n \big|f(x_i) - g(x_i)\big| \text{ and } d_2(f,g) = \Big(\frac{1}{n}\sum_{i=1}^n \big(f(x_i) - g(x_i)\big)^2\Big)^{1/2}.$$

Prove that $D(\mathcal{F}, \varepsilon, d_1) \leq D(\mathcal{F}, \varepsilon, d_2)$.

(5) Consider a class of functions $\mathcal{F} = \{f : \mathcal{X} \to \mathbb{R}\}$ that satisfies the UEC,

$$\forall n \forall x \;\; D(\mathcal{F}, \varepsilon, d_x) \leq \Big(\frac{1}{\varepsilon}\Big)^\alpha.$$

Given $\delta > 0$, consider the following loss function

$$\varphi_\delta(s) = 1 \text{ for } s \leq 0, \varphi_\delta(s) = e^{-\frac{s^2}{\delta^2}} \text{ for } s \geq 0,$$

and consider the class $\varphi_\delta(y\mathcal{F}) = \{\varphi_\delta(yf(x)) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} : f \in \mathcal{F}\}$. Prove that

$$D(\varphi_\delta(y\mathcal{F}), \varepsilon, d_{x,y}) \leq \Big(\frac{\sqrt{2}e^{-1/2}}{\varepsilon\delta}\Big)^\alpha.$$