Let $x \in \mathcal{X}^n$, $x = (x_1, \ldots, x_n)$. Suppose $A \subseteq \mathcal{X}^n$. Define

$$V(A, x) = \{(I(x_1 \neq y_1), \ldots, I(x_n \neq y_n)) : y = (y_1, \ldots, y_n) \in A\},$$

$$U(A, x) = \text{conv } V(A, x)$$

and

$$d(A, x) = \min\{|s|^2 = \sum_{i=1}^n s_i^2, \ s \in U(A, x)\}$$

In the previous lectures, we proved

Theorem 39.1.

$$\mathbb{P}\left(d(A, x) \geq t\right) \leq \frac{1}{\mathbb{P}\left(A\right)} e^{-t/4}.$$

Today, we prove

Theorem 39.2. *The following are equivalent:*

(1) $d(A, x) \leq t$

(2) $\forall \alpha = (\alpha_1, \ldots, \alpha_n), \exists y \in A,$ *s.t.* $\sum_{i=1}^n \alpha_i I(x_i \neq y_i) \leq \sqrt{\sum_{i=1}^n \alpha_i^2 \cdot t}$

Proof. **(1)⇒(2):**

Choose any $\alpha = (\alpha_1, \ldots, \alpha_n)$.

$$\min_{y \in A} \sum_{i=1}^n \alpha_i I(x_i \neq y_i) = \min_{s \in U(A,x)} \sum_{i=1}^n \alpha_i s_i \leq \sum_{i=1}^n \alpha_i s_i^0 \tag{39.1}$$

$$\leq \sqrt{\sum_{i=1}^n \alpha_i^2} \sqrt{\sum_{i=1}^n (s_i^0)^2} \leq \sqrt{\sum_{i=1}^n \alpha_i^2 \cdot t} \tag{39.2}$$

where in the last inequality we used assumption (1). In the above, min is achieved at s^0.

(2)⇒(1):

Let $\alpha = (s_1^0, \ldots, s_n^0)$. There exists $y \in A$ such that

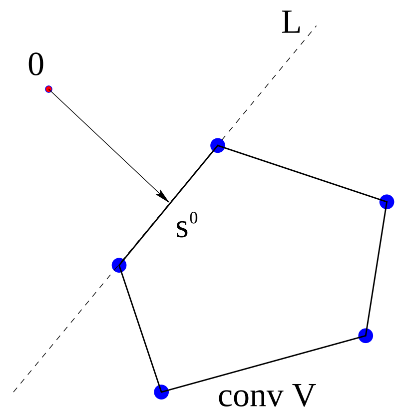
$$\sum_{i=1}^n \alpha_i I(x_i \neq y_i) \leq \sqrt{\sum_{i=1}^n \alpha_i^2 \cdot t}$$

Note that $\sum \alpha_i s_i^0$ is constant on L because s^0 is perpendicular to the face.

$$\sum \alpha_i s_i^0 \leq \sum \alpha_i I(x_i \neq y_i) \leq \sqrt{\sum \alpha_i^2 t}$$

Hence, $\sum (s_i^0)^2 \leq \sqrt{\sum (s_i^0)^2 t}$ and $\sqrt{\sum (s_i^0)^2} \leq \sqrt{t}$. Therefore, $d(A, x) \leq \sum (s_i^0)^2 \leq t$. □

We now turn to an application of the above results: Bin Packing.



Example 39.1. Assume we have $x_1, \ldots, x_n$, $0 \le x_i \le 1$, and let $B(x_1, \ldots, x_n)$ be the smallest number of bins of size 1 needed to pack all $(x_1, \ldots, x_n)$. Let $S_1, \ldots, S_B \subseteq \{1, \ldots, n\}$ such that all x_i with $i \in S_k$ are packed into one bin, $\bigcup S_k = \{1, \ldots, n\}$, $\sum_{i \in S_k} x_i \le 1$.

Lemma 39.1. $B(x_1, \ldots, x_n) \le 2 \sum x_i + 1$.

Proof. For all but one k, $\frac{1}{2} \le \sum_{i \in S_k} x_i$. Otherwise we can combine two bins into one. Hence, $B - 1 \le 2 \sum_k \sum_{i \in S_k} x_i = 2 \sum x_i$ □

Theorem 39.3.

$$\mathbb{P}\left(B(x_1, \ldots, x_n) \le M + 2\sqrt{\sum x_i^2 \cdot t} + 1\right) \ge 1 - 2e^{-t/4}.$$

Proof. Let $A = \{y : B(y_1, \ldots, y_n) \le M\}$, where $\mathbb{P}\left(B \ge M\right) \ge 1/2$, $\mathbb{P}\left(B \le M\right) \ge 1/2$. We proved that

$$\mathbb{P}\left(d(A, x) \ge t\right) \le \frac{1}{\mathbb{P}\left(A\right)} e^{-t/4}.$$

Take x such that $d(A, x) \le t$. Take $\alpha = (x_1, \ldots, x_n)$. Since $d(A, x) \le t$, there exists $y \in A$ such that $\sum x_i I(x_i \ne y_i) \le \sqrt{\sum x_i^2 \cdot t}$.

To pack the set $\{i : x_i = y_i\}$ we need $\le B(y_1, \ldots, y_n) \le M$ bins.

To pack $\{i : x_i \ne y_i\}$:

$$\begin{aligned} B(x_1 I(x_1 \ne y_1), \ldots, x_n I(x_n \ne y_n)) &\le 2 \sum x_i I(x_i \ne y_i) + 1 \\ &\le 2\sqrt{\sum x_i^2 \cdot t} + 1 \end{aligned}$$

by Lemma.

Hence,

$$B(x_1, \ldots, x_n) \le M + 2\sqrt{\sum x_i^2 \cdot t} + 1$$

with probability at least $1 - 2e^{-t/4}$.

By Bernstein's inequality we get

$$\mathbb{P}\left(\sum x_i^2 \leq n\mathbb{E}x_1^2 + \sqrt{n\mathbb{E}x_1^2 \cdot t} + \frac{2}{3}t\right) \geq 1 - e^{-t}.$$

Hence,

$$B(x_1, \ldots, x_n) \lesssim M + 2\sqrt{n\mathbb{E}x_1^2 \cdot t}$$

□