11/29/04

Recall APT from last time:
Let b be an integer and $\beta = \sqrt[3]{b}$ and m, n satisfy
$m+1 \geq \frac{2}{3}n \geq m \geq 3$.
Then $\exists$ a polynomial with integer coefficients
$$F(x,y) = P(x) + y\,Q(x) = \sum_{i=0}^{m+n} u_i X^i + v_i X^i Y \quad \text{s.t.}$$

$$F^{(k)}(\beta,\beta) = 0 \text{ for all } 0 \leq k \leq n, \quad \max_{0 \leq i \leq m+n} \{|u_i|, |v_i|\} \leq 2 \cdot (6b)^{9(m+n)}$$

<u>Smallness Theorem</u>
Let $F(X,Y)$ be APT polynomial. $\exists$ constant $c_1 > 0$, depending only on b, such that $\forall x, y : |x-\beta| \leq 1$ and any integer $0 \leq t \leq n$

$$|F^{(t)}(x,y)| \leq c_1^n \{ |x-\beta|^{n-t} + |y-\beta| \}.$$

At (β,β), partial derivative of F vanish, so
$$F(x,y) = P(x) + Y\,Q(x).$$

$$= \text{(taylor expansion about } (\beta,\beta) \text{)}$$

$$\sum_{k,j \in \mathbb{N}} \frac{1}{k!\,j!} \frac{\partial^{k+j} F}{\partial X^k Y^j}(\beta,\beta) \cdot (X-\beta)^k (Y-\beta)^j$$

$$= \sum_{k=0}^{m+n} F^{(k)}(\beta,\beta)(x-\beta)^k + \sum_{k=0}^{m+n} Q^{(k)}(\beta)(x-\beta)^k(Y-\beta)$$

$F^{(k)}(\beta,\beta) = 0$ for all $0 \leq k \leq n$, so

$$F(X,Y) = \sum_{k=n}^{m+n} F^{(k)}(\beta,\beta)(X-\beta)^k + \sum_{k=0}^{m+n} Q^{(k)}(\beta)(X-\beta)^k(Y-\beta)$$

To find $F^{(t)}(X,Y)$ differentiate t times, then divide by $t!$

$$F^{(t)}(X,Y) = \sum_{k=n}^{m+n} F^{(k)}(\beta,\beta)\binom{k}{t}(X-\beta)^{k-t} + \sum_{k=0}^{m+n} Q^{(k)}(\beta)\binom{k}{t}(X-\beta)^{k-t}(Y-\beta)$$

$$= \left\{\sum_{k=n}^{m+n} F^{(k)}(\beta,\beta)\binom{k}{t}(X-\beta)^{k-n}\right\}(X-\beta)^{n-t} + \left\{\sum_{k=0}^{m+n} Q^{(k)}(\beta)\binom{k}{t}(X-\beta)^{k-t}\right\}(Y-\beta).$$

because $(X-\beta)^{n-t}$ and $(Y-\beta)$ are factors, values of X, Y close to β make $F^{(t)}(X,Y)$ very small.

Let $X=x$ and $Y=y$ and use triangle inequality.

$$|F^{(t)}(x,y)| \le \left\{\sum |F^{(k)}(\beta,\beta)|\binom{k}{t}|x-\beta|^{k-n}\right\}|x-\beta|^{n-t} + \left\{\sum |Q^{(k)}(\beta)|\binom{k}{t}|x-\beta|^{k-t}\right\}|y-\beta|$$

$|x-\beta| \le 1$ by assumption

$\binom{k}{t}|x-\beta|^{e} \le 2^{m+n}$ for any $k \le m+n$, any $e \ge 0$.

$$|F^{(t)}(x,y)| \le C_1\left\{|x-\beta|^{n-t} + |y-\beta|\right\}.$$

$$|F^{(k)}(\beta,\beta)| = \left|\sum_{i=k}^{m+n}\binom{i}{k}\left(u_i\beta^{i-k} + v_i\beta^{i-k+1}\right)\right|$$

$$\le (m+n+1)\cdot \max_{0\le i\le m+n}\binom{i}{k}\cdot 2\max_{0\le i\le m+n}\{|u_i|,|v_i|\}\beta^{m+n}$$

$$k\le m+n.$$

$$\sum_{k=n}^{m+n}|F^{(k)}(\beta,\beta)|\binom{k}{t}|x-\beta|^{k-n} \le (m+1)\cdot 4\left(2^{38}b^{28/3}\right)^{m+n}\cdot 2^{m+n}$$

$m+1 =$ # of terms in the sum.

because $m\le \frac{2}{3}n$

$$\le \left(2^{42}b^{28/3}\right)^{m+n} \le \left(2^{70}b^{140/9}\right)^{n}$$

$$|Q^{(k)}(\beta)| = \left|\sum_{i=k}^{m+n}\binom{i}{k}v_i\beta^{i-k}\right| \le (m+n+1)\,2^{m+n}\cdot \max_{0\le i\le m+n}|v_i|\,\beta^{m+n}$$

$$\le 2^{2(m+n)}\cdot 2\,(16b)^{9(m+n)}\cdot b^{(m+n)/3}$$

$$= 2\left(2^{38}b^{28/3}\right)^{m+n}$$

$$\sum_{k=0}^{m+n}|Q^{(k)}(\beta)|\binom{k}{t}|x-\beta|^{k-t}\le$$

$$(m+n+1)\,2\left(2^{38}b^{28/3}\right)^{m+n}\cdot 2^{m+n}$$

$$\le \left(2^{41}b^{28/3}\right)^{m+n} \le \left(2^{205/3}b^{140/9}\right)^{n}$$

$$|F^{(t)}(x,y)| \le c_1\{|x-\beta|^{n-t}+|y-\beta|\},\quad c_1 = 2^{70}b^{140/9}$$