12/3/04

DAT $b \in \mathbb{Z}^+$ $\beta = \sqrt[3]{b} \notin \mathbb{Z}$.

C constant positive

There are only finitely many integer solutions (p, q)

s.t. $\left| p/q - \beta \right| < \frac{C}{q^2}$. $*$

Proof by contradiction. Assume $\infty \#(p,q)$

C_1, C_2 constants depending only on b.

$* \Rightarrow |p - \beta q| < C$ $\quad q \in \mathbb{Z}$ $q \geq 1$.

(1) $\exists$ some (p_1, q_1) satisfying $q_1 > e^{9C_2}$ $\quad q_1 > (2C_1 C)^{18}$

(p_2, q_2) $\quad q_2 > q_1^{65}$

Choose n to be the integer satisfying

$$n \leq \frac{9}{8} \frac{\log q_2}{\log q_1} < n+1. \qquad (3)$$

$$q_1^{\frac{8}{9}n} \leq q_2 \leq q_1^{\frac{8}{9}(n+1)}$$

(2) $\frac{\log q_2}{\log q_1} > 65.$ $\qquad n > \frac{9}{8}(65-1) = 72.$ $\qquad$ (4)

A.P.T. b, β

$m, n \in \mathbb{Z}$ satisfy $\quad m+1 \geq \frac{2}{3} n \geq m > 3$

$\exists$ poly $F(x,y)$ with int. coeffs s.t.

$$F(x,y) = P(x) + YQ(x) = \sum_{i=1}^{m+n} u_i x^i + v_i x^i y$$

with

$$F^{(k)}(\beta, \beta) = 0 \quad \forall\, 0 < k < n.$$

NVT Let p_1/q_1, $\frac{p_2}{q_2}$ be rational nos in lowest terms.
$\exists$ $C_2 > 0$ depending only on b s.t. there is an integer $0 \le t \le 1 + \frac{C_2 n}{\log q_1}$ with $F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \ne 0$.

Now apply above to our situation. Choose n,
get $F(x,y)$

$$t \le 1 + \frac{C_2 n}{\log q_1} \qquad F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \ne 0 \tag{5}$$

From (1) $$t \le 1 + \frac{1}{9} n \tag{6}$$

deg $F^{(t)}(x,y)$ with respect to x is at most $m+n$
" " y is 1

$$F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) = \frac{\text{integer}}{q_1^{m+n} q_2} \ne 0.$$

Thus $$\left|F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \ge \frac{1}{q_1^{m+n} q_2}$$

APT $\Rightarrow$ $m \ge \frac{2}{3} n$ ~~always~~ also using (3)

$$q_2 < q_1^{8/9(n+1)}$$

$$\left|F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \ge \frac{1}{q^{\frac{23}{9}n + \frac{8}{9}}}.$$

Smallness Theorem.
$\exists$ $C_1 > 0$ C_1 depends only on b s.t. for any real x, y with $|x - \beta| \le 1$, any $0 \le t < n$

$$|F^{(t)}(x,y)| \le C_1^n \{ |x-\beta|^{n-t} + |y-\beta| \}$$

S.T. implies

$$\left|F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| \le C_1^n \left\{ \left|\frac{p_1}{q_1} - \beta\right|^{n-t} + \left|\frac{p_2}{q_2} - \beta\right| \right\}$$

$$\le C_1^n \left\{ \left(\frac{c}{q_1^3}\right)^{n-t} + \frac{c}{q_1^3} \right\}$$

$$\le C_1^n \left\{ \left(\frac{c}{q_1^3}\right)^{\frac{8}{9}n-1} + \frac{c}{q_1^{\frac{8}{9}n}} \right\} \quad \text{from (3) and (6)}$$

$$\le \frac{(2C_1 c)^n}{q_1^{\frac{8}{9}n-3}} \le \frac{1}{q_1^{\frac{47}{18}n-3}} \quad \text{from (1).}$$

will show the 2 bounds are contradictory.

$$\frac{1}{q_1^{\frac{23}{9}n+\frac{8}{9}}} \le \frac{1}{q_1^{\frac{47}{18}n-3}}$$

$$q_1^{\frac{1}{18}n-\frac{35}{9}} \le 1 \qquad \text{but } n \ge 72$$

$$q_1^{\frac{1}{9}} \le 1 \qquad \text{but } q_1 \ge 2 \text{ from (1).}$$

This is our contradiction.

$ax^3 + by^3 = c$ has only finitely many solutions in integers. (depends on DAT)

$\left|\frac{p}{q} - \beta\right| \le \frac{c}{q^3}$ has finitely many solution.

Thue : $\beta \in \mathbb{R}$ the root of an irreducible polynomial
$f(x) \in \mathbb{Q}[x]$ $\deg(f) = d \geq 3$
Let $\varepsilon > 0$, $C > 0$

Then there are only finitely many solutions pairs ~~(x, y)~~
(p, q) s.t. $q > 0$ and $\left| \frac{p}{q} - \beta \right| < \frac{C}{q^{\frac{1}{2}d + 1 + \varepsilon}}$

$\tau(d)$ function of degree d.

As ε when $\left| \frac{p}{q} - \beta \right| \leq \frac{C}{q^{\tau(d) + \varepsilon}}$ has finitely many solutions.

$\tau(d) = d$

Thue $\tau(d) = \frac{1}{2}d + 1$

$2\sqrt{d}$

$\sqrt{2d}$

Roth $\tau(d) = 2$.