1 Continuing...

Theorem. There is an equivalence of categories Γ : (irreducible affine algebraic sets)$\to$(finitely generated integral k-algebras). This is a contravariant equivalence. Recall how Γ works: $\Gamma(\Sigma) = k[x_1, \ldots, x_n]/I(\Sigma)$. If Σ is irreducible then $I(\Sigma)$ is prime so $\Gamma(\Sigma)$ is an integral domain.

Further, we proved that if we have a morphism $\rho : \Sigma_1 \to \Sigma_2$ then there is an induced map $\rho^* : \Gamma(\Sigma_2) \to \Gamma(\Sigma_1)$, defined as follows. For some set $\{f_i\}$ the map ρ will send $(\underline{a}) \to (f_i(\underline{a}))$ and so $\rho^* : x_i \mapsto f_i$. In fact this is independent of the choices of the f_is (more than one set may work, but any set gives rise to the same morphism and the same ρ^*.

We can also get from ρ^* back to ρ by making a map between sets of maximal ideals; $m \mapsto g^{*-1}(m)$. Recall that maximal ideals are in 1-1 correspondence with points in k^n, and this ends up being a morphism from Σ_1 to Σ_2.

Example. Let $k^2 = \mathbb{A}^2 \to \mathbb{A}^1 = k$, defined by $(a, b) \mapsto a$. Then we get a map $k[T] \to k[X, Y]$ defined by $T \mapsto X$. Now, $\Gamma(\mathbb{A}^2) = k$ and the surjection from $k[X, Y] \twoheadrightarrow k$ maps X to a and Y to b. The composition, then, maps T to a, as it should.

Recall that $\Gamma(\Sigma)$ can be thought of as $\mathrm{Hom}(\Sigma, \mathbb{A}^1$. Then if we have a map $g : \Sigma_1 \to \Sigma_2$ there is a map $g^* : \mathrm{Hom}(\Sigma_2, \mathbb{A}^1) \to \mathrm{Hom}(\Sigma_1, \mathbb{A}^1)$ defined as follows. If $f : \Sigma_2 \to \mathbb{A}^1$ then $f \mapsto f \circ g$.

Example. (This equivalence cannot be extended to projective algebraic sets.) Suppose we have a curve $C \subset \mathbb{P}^2$ which is a conic, and we have a point $P_0 \in C$ then we get a bijection $\mathbb{P}^1(k) \to C$ which maps lines in $\mathbb{P}^2$ through P_0 to points in C, by mapping a line to the point it intersects C at other than P_0. We decided last week that this should be a morphism.

For example, $C : xz - y^2 = 0$, $P_0 = (0 : 0 : 1) \in C$. We also get a map $P \mapsto$ the line between P and P_0, which maps $C \to \mathbb{P}^1$. This works out as follows: $Q \in C - P_0$ maps $(a : b : c) \mapsto (a : b)$ and $C - (1 : 0 : 0)$ maps $(a : b : c) \mapsto (b : c)$. However, there is no map we can choose $k[s, t] \to k[x, y, z]/(xz - y^2)$ that induces these. This is because if we have the ideal $(x - a, y - b, z - c)$ we should (by the first type) get $(s - a, t - b)$ but we should also get $(s - b, t - c)$, either of which is no good because they don't agree.

However, we will be able to look at pieces and construct this piecewise.

2 Topological Diversion

Goal: define a topology on affine sets and define sheaves.

Def. The Zariski toplogy on $\mathbb{A}^n$ is the one for which closed sets are $V(A)$ with $A \subset k[x_1, \ldots, x_n]$ an ideal. The intersection of arbitrary numbers of closed sets is still closed because this gives the sum of all the ideals, which is still an ideal. Further, the union of two closed sets is closed, and the whole space is closed ($A = (0)$) and the empty set is closed (A is the whole ring).

Ex. What are the open sets in $\mathbb{A}^1$? $k[X]$ is a PID so closed sets are $V(f) : f \in k[x]$. Thus, open subsets of $\mathbb{A}^1$ are $\mathbb{A}^1 - \{p_1, \ldots, p_r\}$. Thus, any two open sets intersect! This indicates that the space is not Hausdorff.

Def. If $\Sigma \subset \mathbb{A}^n$ is an affine algebraic set, then the Zariski topology is the induced topology via intersection with Σ.

Equivalently, a closed set in Σ is $V(A) \cap \Sigma \iff$ ideals in $\Gamma(\Sigma)$. The idea is that we have $A \subset k[x_1, \ldots, x_n] \twoheadrightarrow \Gamma(\Sigma) \xrightarrow{p} k$. We can also think of A as mapping onto an ideal $\overline{A} \hookrightarrow \Gamma(\Sigma)$ which maps to zero under this induced map whenever $p \in V(A) \cap \Sigma$. (?)

So closed sets in $\Sigma = V(B) \leftrightarrow$ intersections $V(A) \cap V(B) = V(A + B)$ These in turn are in association with ideals in $k[x_1, \ldots, x_n]$ containing B, which in turn are those ideals in the quotient $k[x_1, \ldots, x_n]/B$. (This makes more sense to me.)

Ex. Any map? $k \to k$ is continuous. ("Does anyone see why this is clear?") Scratch that. Any bijection is continuous, since the preimage of a finite set is finite and the preimage of a cofinite set is cofinite. However, not all bijections are polynomials; in fact, only linear polynomials can be bijections.

Lemma. Any morphism $f : \Sigma_1 \to \Sigma_2$ is continuous.

Pf. We need that $f^{-1}(U) = V$ where if U is a closed set then V is closed. We know that $f^{-1}(V(A)) = V(f^*(A))$, so this is continuous. (Check this last statement... not the implication, I get that part.)

Def. A top. space X is Noetherian if its closed sets satisfy the descending chain condition (DCC), meaning, if we have a sequence $X \supset Z_0 \supset Z_1 \supset \ldots$ then eventually $Z_n = Z_m$ for all $m > n$.

Lemma. Affine algebraic sets with the Zariski topology are Noetherian.

Pf. If we have a descending chain, we get an *ascending* chain of ideals $I_0 \subset I_1 \subset \ldots \subset \Gamma(\Sigma)$. Luckily, $\Gamma(\Sigma)$ is an f.g. k-algebra, so it is a Noetherian ring, thus it satisfies the *ascending* chain condition, so eventually, $I_n = I_m$ for all $m > n$.

Ex. Any open set of an algebraic set is quasi-compact (meaning every open cover has a finite subcover: compact usually means this plus Hausdorff). (WEIRD: usually closed things are compact, not open things.) We will prove this in the homework.

Def. A closed set in $¶^n$ is $V(A)$ where A is a homogenous ideal. This is the Zariski toplogy on $¶^n$.

Lemma. The standard $U_i \subset ¶^n$ are open.

Pf. $U_i = \overline{V(X_i)}$. QED.

Cor. (to the example) Any projective algebraic set is noetherian.

Pf. Say we have a descending chain. We know that $\Sigma \cap U_i$ is noetherian. We thus know that when we intersect our chain with U_i we do get stabilization. Thus, we choose a point n which is the largest of all the n_i for which the chain stabilizes when intersected with U_i, and the chain will stabilize there. Thus, Σ is noetherian.

3 Sheaves

Let X be a topological space. Consider continuous maps $X \xrightarrow{f} \mathbb{R}$. We can define f locally by $f_i : U_i \to \mathbb{R}$ where $\{U_i\}$ is an open covering of X, so that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ (that is, they agree on overlaps).

Def. A *presheaf* F on X is a contravariant functor $F : Op(X) \to Set$. ($Op(X)$ is the category of open sets of a topological space. The objects are open sets and the maps are inclusion maps if such a map exists. Set is the category of sets; the objects are sets and the maps are set maps.

Equivalently, a presheaf consists of:

- $\forall U \subset X$ open, a set $F(U)$.
- $\forall$ inclusions $V \subset U$, a map $res_{UV} : F(U) \to F(V)$.
- $res_{UU} = id_{F(U)}$.
- Given $W \subset V \subset U$ the diagram commutes (ie, $res_{UW} = res_{VW} \circ res_{UV}$).

Note that instead of using Set we could use whatever category we want and the definition still makes sense.

Ex. Consider $F(U) = \{$ continuous maps $U \to \mathbb{R}\}$. This is a presheaf since we can use $res_{UV} : f \mapsto f|_V$.

Ex. $F(U) = G$ for G some fixed group. If $V \subset U$ then use the identity map. This is a (very trivial) presheaf of groups.

Note: for the previous example, if given $f_i \in F(U_i)$ where $\{U_i\}$ covers X such that $res_{U_i U_i \cap U_j} = res_{U_j U_i \cap U_j}$ in $F(U_i \cap U_j)$, then we get $f \in F(X)$ so that $f_i = res_{X U_i}(f)$: that is, gluing works. This fails in the latter example. For instance, $X = \{a, b\}$ with the discrete topology. Let $g_1 \in F(A)$ and $g_2 \in F(B)$ where $A = \{a\}, B = \{b\}$. Then, if $g_1 \neq g_2$ there is no way to make them agree over the whole of X.

Def. Roughly, a *sheaf* is a presheaf where we can glue properly. More specifically, a sheaf is a presheaf such that

1. Elements are determined by their restriction to an open cover, and
2. We should be able to glue local data.

We will be more specific next time.