November 4, 2003

1 Completeness

Def. Let X be a variety. We say X is *complete* (proper) if for all Y the map $X \times Y \to Y$ is closed.

Ex. $\mathbb{A}^n$ is not complete.

Lemma. (1) If X and Y are complete, $X \times Y$ is complete. (2) If X is complete, $Z \subset X$ closed subvariety, then Z is complete. (3) An affine variety X is complete $\iff$ $X = (*, k)$.

Pf. (1): Let Z be a variety, we want $X \times Y \times Z \to Z$ to be closed. But this map is just the composition $X \times Y \times Z \to Y \times Z \to Z$, both of which are closed, so the map we want to be closed is closed.

(2): Let Y be a variety, consider the map $Z \times Y \hookrightarrow X \times Y \to Y$. The first of these is closed (follows from the topology on products) and the second is closed by completeness of X.

Theorem 1. Any projective variety is complete.

Theorem 2. (Chow's Lemma). If X is a complete variety, then there is a projective variety Y and a surjective birational map $\pi : Y \to X$.

We will get to the proof of these, hopefully today.

Lemma. If $f : X \to Y$ is a morphism of varieties and X is complete, then $f(X) \subset Y$ is closed.

Proof. $f(x) = p_2(\Gamma_f(X))$ where Γ_f is the graph $X \xrightarrow{\Gamma_f} X \times Y$ where $x \mapsto (x, f(x))$. We know this map is closed; it was part of the definition of varieties. Thus, $\Gamma_f(X)$ is closed, and p_2 is closed mapping $X \times Y \to Y$ since X is complete, and so $f(X)$ is closed.

Completeness actually corresponds to the ability to complete maps to limit points. I.E. suppose we had a map $[0, 1) \to X$, then we could complete the map to find one $[0, 1] \to X$ that agrees.

Olsson takes an aside about the functor h_X. Basically, the above interpretation corresponds to the idea that $h_X(C) \to h_X(U)$ is surjective, where C is dimension 1 and $U \hookrightarrow C$ where every local ring in C is regular (ie, $C = [0, 1]$ and $U = [0, 1)$.)

Pf. of Theorem 1.

Let Y be a variety. We want $\mathbb{P}^n \times Y \to Y$ closed. We can assume Y is affine. In fact we can assume Y is $\mathbb{A}^r$ because we have the diagram

$$\begin{array}{ccc} \mathbb{P}^n \times Y & \to & Y \\ \downarrow & & \downarrow \\ \mathbb{P}^n \times \mathbb{A}^r & \to & \mathbb{A}^r \end{array}$$

Let $R = \Gamma(\mathbb{A}^r, O_{\mathbb{A}^r})$, and look at $S = R[X_0, \ldots, X_n]$. This is a graded ring. Closed sets in $\mathbb{P}^n \times \mathbb{A}^r$ correspond to homogeneous ideals in S. To see this, Let $U_i = U_{x_i \neq 0} \times \mathbb{A}^r$. We have $\Gamma(U_i, O_{U_i}) = k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \otimes_k R = (S_{x_i})_{(0)}$, the degree 0 part of S localized at x_i. Now think about the intersection of any closed set with the open cover, and we get a homogeneous ideal.

Note here: it really isn't doing us any good here to be using $\mathbb{A}^r$ instead of an arbitrary affine variety.

Let $I(Z)$ be the ideal generated by homog. $f \in S$ such that $f(Z) = 0$.

Lemma. For all i, $I(Z \cap U_i)$ is generated by the image of $(I(Z)_{X_i})_{(0)} \to I(Z \cap U_i)$.

Pf. of lemma. Let $g \in I(Z \cap U_i) \subset R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. As an element in this ring, it has some denominator, so we can multiply by some power of x_i to get $x_i^m g \in R[x_0, \ldots, x_n]$. This may not vanish everywhere on Z, for instance on $Z \cap (U_i^C) = Z \cap V(x_i)$, because when $x_i = 0$ this may not work, so we just multiply by one more x_i to get $x_i^{m+1} g$ and this now vanishes on $Z \cap U_i$.

So we have a homoegenous $A \subset S, V(A) = Z$, and we have our map $\mathbb{P}^n \times \mathbb{A}^n \to \mathbb{A}^n$. Choose $y \notin p_2(Z)$. We will show there is an open set including y which has empty intersection with $p_2(Z)$; this will prove that $p_2(Z)^C$ is open, and thus prove $p_2(Z)$ is closed.

Let $m \subset R$ be the maximal ideal of y. Consider $Z \cap U_i \hookrightarrow U_{x_i \neq 0} \times Y \hookleftarrow U_{x_i \neq 0} \times \{y\}$. Both are closed maps, and they do not intersect.

We have $m \hookrightarrow R \to R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]/I(Z \cap U_i)$. We know m maps to the ideal (1) since this corresponds to the maps above, in which the intersection is empty. Thus, $1 = a_i + \sum_j m_{i,j} g_{i,j}$ where $a_i \in I(Z \cap U_i)$ and $m_{i,j} \in m$ in $R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. Thus, there is some N_i such that $x_i^{N_i} = a_i' + \sum_j m_{i,j} g_{i,j}'$ where $a_i' \in I(Z)$ and this equation holds in S.

Look at the degree N_i piece $S_{N_i} = (R[X_0, \ldots, X_n])_{N_i}$, So we get $S_N = A_N + mS_N$ for some big enough N. Thus, $M = S_N/A_N$ is a f.g. module over R, and $mM = M$ so there is an $f \in R - m$ such that $fS_N \subset A_N$ by Nakayama's lemma.

This f is the one we want. Consider $D(f)$. $D(f) \cap p_2(Z) = \emptyset$. This is because $fS_N \subset A_N \Rightarrow p_2(V(A)) \subset V(f) \subset Y$. This completes the proof of Theorem 1.

Prop. $f : \mathbb{P}^n \to \mathbb{P}^m$, $W = f(\mathbb{P}^n)$ is a variety. Then either $\dim W = n$ or $\dim W = 0$. It may not be an isomorphism: We had our map $t \mapsto t^2, t^3 - t^2$ (I think? The graph he drew looks like an α) and the cusp has two preimages.

Pf. Let $r = \dim W$, assume $1 \leq r \leq n-1$. We know there is a list $f_1, \ldots, f_{r+1} \in k[X_0, \ldots, X_m]$ such that $W \cap V((f_1, \ldots, f_{r+1})) = \emptyset$ and $W \cap V(f_i) \neq \emptyset$. (This is applying that we can keep intersecting with $V(f_i)$, each time reducing the dimension by 1, until we get down to the emptyset.)

Let $Z_i = f^{-1}(V(f_i))$. We know $Z_1 \cap \ldots \cap Z_{r+1} = \emptyset$. There are two possibilities. Either $Z_i = \mathbb{P}^n$ or they're a hypersurface. We know $r + 1 \leq n$, but this can't happen! We proved this before: the intersection of $\leq n$ hypersurfaces is nonempty in $\mathbb{P}^n$.

Thus each $Z_i = \mathbb{P}^n$, but then their intersection can't be empty because there is at least 1 of them ($r \geq 1$). Thus, we have a contradiction, so the dimension of W is either 0 or n.

2 Complex Topology

Martin wants to talk about complex topology for a bit, and then about curves.

Let $X \subset \mathbb{A}^n_{\mathbb{C}}$. What is the complex topology? Take the usual topology on $\mathbb{C}^n$ and give X the induced topology. There is a sheaf of rings O_X on X here; we define this as $O_{\mathbb{C}^n}(U) = \{$ holomorphic functions $U \to \mathbb{C}\}$.

If $V \subset X_{an}$ is open, then define $O_{X_{an}}(V) = \{f : V \to \mathbb{C}$ such that $\forall v \in V$ there is a neighborhood U of v in $\mathbb{A}^n$ and $\tilde{f} \in O_{\mathbb{C}^n}(U)$ restricting to $f\}$. NOTE: X_{an} is basically X under the induced topology from $\mathbb{C}^n$.

Def. An *analytic space* is a pair (X, O_X) where X is a top. space, O_X is a sheaf of functions $X \to \mathbb{C}$ such that there is a finite open cover U_i of X such that $(U_i, O_X|_{U_i})$

Whoops, out of time.