HOMEWORK 3 FOR 18.725, FALL 2015
DUE TUESDAY, SEPTEMBER 29 BY 1PM.

(1) Let Z be an irreducible closed subset in an algebraic variety X. Show that if $\dim(Z) = \dim(X)$ then Z is a component of X.

(2) Let Y be a closed subvariety of dimension r in $\mathbb{P}^n$.

   (a) Suppose that Y can be presented as the set of common zeroes of q homogeneous polynomials. Show that $r \geq n - q$.
   If Y can be presented as the set of common zeroes of q homogeneous polynomials with $q = n - r$ we say that Y is a *set-theoretic complete intersection.*
   If moreover the ideal I_Y can be generated by $n - r$ homogeneous polynomials, then Y is called a *(strict) complete intersection.*

   (b) Show that every irreducible closed subvariety in $\mathbb{P}^n$ is a component in a set theoretic complete intersection of the same dimension.
   [Hint: use induction to construct homogeneous polynomials $P_1, P_2, \ldots, P_{n-r}$, such that the set of common zeroes of $P_1, \ldots P_i$ has dimension $n - i$ and contains our subvariety].

   (c) Show that the twisted cubic curve in $\mathbb{P}^3$ (see problem 2 of problem set 2) is a set theoretic complete intersection.

   (d) (Optional bonus problem) Show that the twisted cubic curve in $\mathbb{P}^3$ is not a strict complete intersection.

(3) Let C be a curve in $\mathbb{P}^2$, x be a point in C and L a line passing through x. Let m be the multiplicity of C at x and M the multiplicity of intersection of C and L at x. Show that $m \leq M$ and that for given C, x the equality $m = M$ holds for all but finitely many lines L as above.

(4) Prove Bezout Theorem for two curves of degrees d_1, d_2 in $\mathbb{P}^2$ with no common components

   (a) Assuming $d_1 = 1$.

   (b) Assuming $d_1 = 2$ and the first curve is irreducible; you can also assume that characteristic of the base field is different from two.

   [Hint: first show that in a special case the multiplicity of intersection of two curves can be interpreted as follows. Assume that the first curve X is isomorphic to $\mathbb{A}^1$ and let $f : \mathbb{A}^1 \to X$ be the isomorphism. Let P be the equation of the second curve Y. Then the multiplicity of intersection of X and Y at $x = f(a)$ is the multiplicity of a as a root of the polynomial in one variable $Q(t) = P(f(t))$. Now use the isomorphism of the first curve with $\mathbb{P}^1$, choose coordinates so that the infinite line does not contain intersection points and recall a familiar fact about polynomials in one variable].

(5) (Optional bonus problem) Recall from the lecture that Grassmannian $Gr(2, 4)$ is isomorphic to a quadric in $\mathbb{P}^5$. Use this to show that given four lines in $\mathbb{P}^3_k$, the number of lines intersecting each of the four lines is either infinite or equal to one or two.

[Hint: Check that the for a line $L \subset \mathbb{P}^3$ the set of lines intersecting L is parametrized by $Gr(2,4) \cap H$ for a hyperplane $H \subset \mathbb{P}^5$, thus the answer is the number of points in the intersection $L \cap Gr(2,4)$ where $L \subset \mathbb{P}^5$ is a linear subspace of dimension one or higher. Check that the intersection is infinite unless L is a line and refer to problem 3(a) from problem set 2].

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