18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 13

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1. Classification of Ruled Surfaces (contd.)

Recall from last time that we had $(K \cdot D) < 0$ for some effective divisor D, and we wanted to show that X was ruled.

- Step 1: There is an ample H s.t. $(K \cdot H) < 0$, so $|nK| \neq \varnothing$ for all $n \geq 1$.
- Step 2: If $(K^2) > 0$, X is rational and thus ruled.
- Step 3: Otherwise, for all n, there is an effective divisor D on X s.t. $|K + D| = \varnothing$ and dim $|D| \geq n$.
- Step 4: If $|K + D| = \varnothing$ for D effective, then there is a natural, surjective map $\operatorname{Pic}^0_X \to \operatorname{Pic}^0_D$.
- Step 5: If D is an effective divisor s.t. $|K + D| = \varnothing$ and if $D = \sum n_i E_i$, then
  1. All the E_i are nonsingular, and $\sum p_a(E_i) \leq q = h^1(X, \mathcal{O}_X)$.
  2. $\{E_i\}$ is a configuration of curves with no loops, and E_i intersect transversely.
  3. If $n_i \geq 2$, then either
     - (a) E_i is rational,
     - (b) $(E_i)^2 < 0$, or
     - (c) E_i is an elliptic curve with $E_i^2 = 0$ and the normal bundle of E_i in X is nontrivial.

Proof. (1) Since $|K + E_i| \subset |K + D| = \varnothing$, we see that $\operatorname{Pic}^0_X \to \operatorname{Pic}^0_D$ is surjective, and so $\operatorname{Pic}^0_{E_i}$ is an abelian variety and E_i is nonsingular: if it were singular, and $\tilde{E}$ were the normalization of E, $\operatorname{Pic}^0_{E_i}$ would be an extension of $\operatorname{Pic}^0_{\tilde{E}}$ by a nontrivial affine subgroup, i.e. a combination of additive groups $\mathbb{G}_a$ and/or multiplicative groups $\mathbb{G}_m$. See Serre's Algebraic Groups and Flass fields, Oort's "A construction of generalized Jacobian varieties by group extension".) Similarly, $|K + D'| \subset |K + D| = \varnothing$, where $D' = \sum E_i$, so $\operatorname{Pic}^0_{D'}$ is an abelian variety. Thus, any components E_i and E_j of D with $i \neq j$ and $E_i \cap E_j \neq \varnothing$ intersect transversely (else get subgroups isomorphic to $\mathbb{G}_m$ inside $\operatorname{Pic}^0_{D'}$). The exact sequence

$$0 \to \mathcal{O}_{D'} \to \prod \mathcal{O}_{E_i} \to \operatorname{cok} \to 0 \tag{1}$$

gives $h^1(\mathcal{O}_{D'}) \geq \sum h^i(\mathcal{O}_{E_i}) = \sum p_a(E_i)$ because cok is supported in dimension 0. Since $|K+D'| = \varnothing, H^2(\mathcal{O}_X(-D')) = 0$, so $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_{D'})$ is surjective and $\sum p_a(E_i) \leq \dim H^1(\mathcal{O}_X) = q$.

(2) If D contains a loop, $\mathrm{Pic}\,(D)$ contains a subgroup isomorphic to $\mathbb{G}_m$, which is not possible since $\mathrm{Pic}\,^0_D$ is an abelian variety.

(3) If $n_i \geq 2$, then $|K+2E_i| \subset |K+D| = \varnothing$ so $\mathrm{Pic}\,^0_{2E_i}$ is an abelian variety. Then the natural map $\mathrm{Pic}\,^0_{2E_i} \to \mathrm{Pic}\,^0_{E_i}$ is an isogeny of abelian varieties, as follows: consider the exact sequences (see lemma below)

$$\begin{aligned} 0 \to I = \mathcal{O}_{E_i}(-E_i|_{E_i}) \to \mathcal{O}_{2E_i} \to \mathcal{O}_{E_i} \to 0 \\ 0 \to I \xrightarrow{\alpha} \mathcal{O}^*_{2E_i} \to \mathcal{O}^*_{E_i} \to 1 \end{aligned} \tag{2}$$

$(I^2 = 0, \alpha(x) = 1 + x)$. The second sequence induces

(3)
$$0 \to H^1(I) \to \mathrm{Pic}\,(2E_i) \to \mathrm{Pic}\,(E_i) \implies 0 \to H^1(I) \to \mathrm{Pic}\,^0(2E_i) \to \mathrm{Pic}\,^0(E_i)$$

If $H^1(I)$ is nonzero, the abelian variety $\mathrm{Pic}\,^0_{2E_i}$ would contain a copy of $\mathbb{G}_a$ which is impossible. So $H^1(E) = 0$. Then the cohomology of the first exact sequence gives $0 = H^1(I) \to H^1(\mathcal{O}_{2E_i}) \to H^1(\mathcal{O}_{E_i}) \to 0$ ($H^2 = 0$ since the sheaf I is supported on a curve). So the tangent map of $\mathrm{Pic}\,^0_{2E_i} \to \mathrm{Pic}\,^0_{E_i}$ is an isomorphism and the map $\mathrm{Pic}\,^0_{2E_i} \to \mathrm{Pic}\,^0_{E_i}$ is an isogeny. In particular, $H^1(\mathcal{O}_{E_i}(-E_i|_{E_i})) = 0$. If $(E_i)^2 \geq 0$ then since $\deg(\mathcal{O}_{E_i}(-E_i|_{E_i})) = -(E_i)^2 \leq 0$. Using RR, we see that either E_i is a rational curve (in which case $(E_i)^2 = 0$ or 1) or an elliptic curve (in which case $(E_i)^2 = 0$, and $\mathcal{O}_{E_i}(-E_i|_{E_i})$ is not isomorphic to $\mathcal{O}_{E_i}$ and the normal bundle is nontrivial). □

Lemma 1. *If $D = D' + D''$ are effective divisors on X $(D', D'' > 0)$ then there is an exact sequence $0 \to \mathcal{O}_{D''}(-D') \to \mathcal{O}_D \to \mathcal{O}_{D'} \to 0$ where $\mathcal{O}_{D''}(-D') = \mathcal{O}_X(-D') \otimes \mathcal{O}_{D''}$.*

Proof. Tensoring $0 \to \mathcal{O}_X(-D'') \to \mathcal{O}_X \to \mathcal{O}_{D''} \to 0$ by $\mathcal{O}_X(-D')$ gives $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X(-D') \to \mathcal{O}_{D''}(-D') \to 0$. Now, $\mathcal{O}_X(-D) \subset \mathcal{O}_X(-D')$ and $\mathcal{O}_D \to \mathcal{O}'_D$ is surjective.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-D) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(-D') & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{D'} & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & \mathcal{O}_{D''}(-D') & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array} \tag{4}$$

Using the snake lemma proves the result. □

We now conclude the proof of classification. Let X be a surface, $x_1, \dots, x_n$ distinct closed points on X, D a divisor s.t. $\dim |D| \geq 3n$.

Lemma 2. *There is an effective divisor $D' \in |D|$ s.t. every x_i is on D' as a multiple point.*

Proof. As before, we have an exact sequence of abelian sheaves

$$0 \to \mathcal{O}_X(D) \otimes I_1 \otimes \cdots \otimes I_n \to \mathcal{O}_X(D) \to \bigoplus k(x_i)^3 \to 0 \tag{5}$$

where I_i is the kernel of $\mathcal{O}_X \to \mathcal{O}_{X,x_i}/\mathfrak{m}^2_{X,x_i}$: this follows from the fact that $\dim \mathcal{O}_{X,x_i}/m^2_{X,x_i} = 3$, so $\mathcal{O}_{X,x_i}/m^2_{x,x_i} \cong k(x_i)^3$ as a skyscraper abelian sheaf supported at x_i. Taking cohomology gives

$$0 \to H^0(\mathcal{O}_X(D) \otimes I_1 \otimes \cdots \otimes I_n) \to H^0(\mathcal{O}_X(D)) \to \bigoplus k(x_i^3) \to \cdots \tag{6}$$

where the dimension of the second term is $\geq 3n+1$ and that of the third term is $3n$. Thus, we obtain a nonzero section of $\mathcal{O}_X(D) \otimes I_1 \otimes \cdots \otimes I_n$: taking its divisor of zeros gives the desired divisor. □

Now, if $q = h^1(X, \mathcal{O}_X) = 0$, then since $p_2 = 0$, we get X rational and thus ruled. So we assume that $q > 0$.

Proposition 1. *Through every point $x \in X$ there is a nonsingular rational curve on X.*

Proof. If not, then we claim that there are only finitely many smooth rational curves on X. Indeed, let $f : X \to B$ be obtained from the Albanese morphism $\alpha : X \to \mathrm{Alb}\,(X)$ by Stein factorization (so B is the normalization of $\alpha(X)$ in the rational function field $k(X)$). If there were infinitely many nonsingular rational curves, then each of them would be contained in a fiber of f (since their image in $\mathrm{Alb}\,(X)$ is a single point). So there are infinitely many points $b \in B$ s.t. $\dim\,(f^{-1}(b) = 1$, so B cannot be a surface and is instead a nonsingular curve. Since the general fiber of f is integral, and since infinitely many fibers contain at least one nonsingular rational curve, at least one of these closed fibers must be a smooth rational curve. Then the proof of Tsen's theorem tells us that all the fibers of f are smooth rational curves, contradicting the hypothesis. Therefore, there exist only finitely many rational curves on X.

Now fix a projective embedding $X \to \mathbb{P}^n$. For a fixed integer $d \geq 1$, there are only finitely many integral curves E on X with $\deg\,(E) = d$ and $H^0(N_E) = 0$ where N_E is the normal bundle of E in X. [Since if $\mathcal{C}(d)$ is the Hilbert scheme of curves of degree d on X, $e \in \mathcal{C}(d)$ the closed point corresponding to E, then the tangent space to $\mathcal{C}(d)$ at e is $H^0(N_E) = 0$ and thus e is isolated. Since the Hilbert scheme is projective, there are only finitely many isolated points]. In particular, there are only finitely many curves of degree d satisfying either

$(E^2) < 0$ or E elliptic and N_E not isomorphic to $\mathcal{O}_E$. Let $\mathcal{F}$ be the family of all nonsingular curves E on X that are either rational or of the above types. $\mathcal{F}$ is countable. Assume that k is uncountable (postpone the countable case for the moment). A general hyperplane H on $\mathbb{P}^n$ intersects X in a nonsingular connected curve C with $C \notin \mathcal{F}$ (Bertini, $\mathcal{F}$ has only finitely many of degree $= \deg(X)$). $C \smallsetminus \bigcup_{E \in \mathcal{F}}(C \cap E)$ is infinite, so $X \smallsetminus \bigcup_{E \in \mathcal{F}} E$ is infinite.

Let $x_1, \ldots, x_q$ be q distinct points in $X \smallsetminus \bigcup E$. By the previous steps, there is an effective divisor D' s.t. $(K \cdot D') < 0, |K + D'| = \varnothing, \dim |D'| \geq 3q$. Then by the lemma, $\exists$ a divisor $D = \sum n_i E_i$ in $|D'|$ s.t. $x_1, \ldots, x_q$ are multiple points of D. If E_i is a component of D that passes through x_j, then $E_i \notin \mathcal{F}$ (since $x_j \in X \smallsetminus \bigcup E$). So $n_j \geq 2$ is not possible, i.e. $n_j = 1$. We also know that E_i is nonsingular. Since x_j is a multiple point of D_j, it follows that at least 2 components of D must pass through x_j. Since the intersection graph of $\{E_i\}$ has no loops (i.e. it's a tree), there must be at least $q+1$ such E_i. Also, $p_a(E_i) \geq 1$ since $\mathcal{F}$ includes all nonsingular rational curves, $\sum p_a(E_i) \geq q+1$. This contradicts $\sum P_a(E_i) \leq q$.

Therefore $\exists$ a nonsingular rational curve through any point of X. Let $\alpha : X \to \operatorname{Alb}(X)$ be the Albanese morphism. It follows as above that for any rational curve C on $\alpha(C)$ is a point $\implies \dim \alpha(X) = 1$. Let $X \to B$ be the Stein factorization. The general fiber is integral, and the special fibers contain smooth rational curves $\implies$ argument above shows that $X \to B$ is a ruling. This proves the theorem when k is uncountable.

Now, if k is countable, let $k \hookrightarrow k'$ be an extension with k' algebraically closed and uncountable. Then $X' = X \times_k k'$ satisfies the same properties (e.g. $\exists$ an effective divisor D' s.t. $K_{X'} \cdot D < 0$), implying that X' is ruled over B', obtained by Stein factorization for $\alpha : X' \to \alpha(X')$ (α is the Albanese morphism). By functoriality, $B' = B \times_k k'$, and it is easy to check that X is ruled over B. □