THE MODULI SPACE OF CURVES

1. The moduli space of curves and a few remarks about its construction

The theory of smooth algebraic curves lies at the intersection of many branches of mathematics. A smooth complex curve may be considered as a Riemann surface. When the genus of the curve is at least 2, then it may also be considered as a hyperbolic two manifold, that is a surface with a metric of constant negative curvature. Each of these points of view enhance our understanding of the classification of smooth complex curves. While we will begin with an algebraic treatment of the problem, we will later use insights offered by these other perspectives.

As a first approximation we would like to understand the functor

$$\mathcal{M}_g : \{\text{Schemes}\} \to \{\text{sets}\}$$

that assigns to a scheme Z the set of families (up to isomorphism) $X \to Z$ flat over Z whose geometric fibers are smooth curves of genus g.

There are two problems with this functor. First, there does not exist a scheme that represents this functor. Recall that given a contravariant functor F from schemes over S to sets, we say that a scheme $X(F)$ over S and an element $U(F) \in F(X(F))$ represents the functor finely if for every S scheme Y the map

$$\text{Hom}_S(Y, X(F)) \to F(Y)$$

given by $g \to g^*U(F)$ is an isomorphism.

Example 1.1. The main obstruction to the representability (in particular, to the existence of a universal family) of $\mathcal{M}_g$ is curves with automorphisms. For instance, fix a hyperelliptic curve C of genus g. Let τ denote the hyperelliptic involution of C. Let S be a $K3$-surface with a fixed point free involution i such that S/i is an Enriques surface E. To be very concrete let C be the normalization of the plane curve defined by the equation $y^2 = p(x)$ where $p(x)$ is a polynomial of degree $2g+2$ with no repeated roots. The hyperelliptic involution is given by $(x, y) \mapsto (x, -y)$. Let Q_1, Q_2, Q_3 be three general ternary quadratic forms. Let the $K3$-surface S be defined by the vanishing of the three polynomials $Q_i(x_0, x_1, x_2) + Q_i(x_3, x_4, x_5) = 0$ with the involution that exchanges the triple (x_0, x_1, x_2) with (x_3, x_4, x_5). Consider the quotient of $C \times S$ by the fixed-point free involution $\tau \times i$. The quotient is a non-trivial family over the Enriques surface E; however, every fiber is isomorphic to C. If $\mathcal{M}_g$ were finely represented by a scheme, then this family would correspond to a morphism from E to it. However, this morphism would have to be constant since the moduli of the fibers is constant. The trivial family would also give rise to the constant family. Hence, $\mathcal{M}_g$ cannot be finely represented.

There are two ways to remedy this problem. The first way is to ask a scheme to only coarsely represent the functor. Recall the following definition:

Definition 1.2. Given a contravariant functor F from schemes over S to sets, we say that a scheme $X(F)$ over S coarsely represents the functor F if there is a natural transformation of functors $\Phi : F \to \mathrm{Hom}_S(*, X(F))$ such that

(1) $\Phi(spec(k)) : F(spec(k)) \to \mathrm{Hom}_S(spec(k), X(F))$ is a bijection for every algebraically closed field k,
(2) For any S-scheme Y and any natural transformation $\Psi : F \to \mathrm{Hom}_S(*, Y)$, there is a unique natural transformation
$$\Pi : \mathrm{Hom}_S(*, X(F)) \to \mathrm{Hom}_S(*, Y)$$
such that $\Psi = \Pi \circ \Phi$.

The main theorem of moduli theory asserts that there exists a quasi-projective moduli scheme coarsely representing the functor $\mathcal{M}_g$.

Alternatively, we can ask for a Deligne-Mumford stack that parameterizes smooth curves. Below we will give a few details explaining how both constructions work.

There is another serious problem with the functor $\mathcal{M}_g$. Most families of curves in projective space specialize to singular curves. This makes it seem unlikely that any moduli space of smooth curves will be proper. This, of course, is in no way conclusive. It is useful to keep the following cautionary tale in mind.

Example 1.3. Consider a general pencil of smooth quartic plane curves specializing to a double conic. To be explicit fix a general, smooth quartic F in $\mathbb{P}^2$. Let Q be a general conic. Consider the family of curves in $\mathbb{P}^2$ given by
$$C_t : Q^2 + tF.$$
I claim that after a base change of order 2, the central fiber of this family may be replaced by a smooth, hyperelliptic curve of genus 3. The total space of this family is singular at the 8 points of intersection of Q and F. These are ordinary double points of the surface. We can resolve these singularities by blowing up these points.

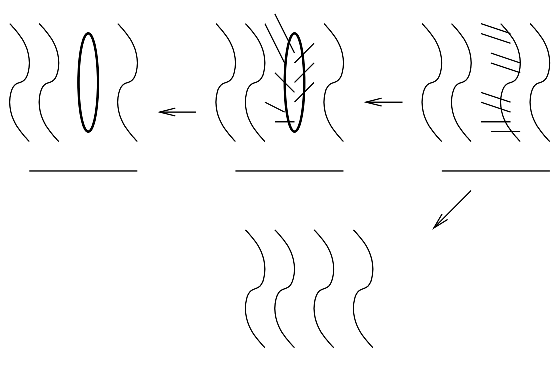

Figure 1. Quartics specializing to a double conic.

We now make a base change of order 2. This is obtained by taking a double cover branched at the exceptional curves $E_1, \dots, E_8$. The inverse image of the proper transform of C_0 is a double cover of $\mathbb{P}^1$ branched at the 8 points. In particular, it is a hyperelliptic curve of genus 3. The inverse image of each exceptional curve is rational curve with self-intersection -1. These can be blown-down. Thus, after base change, we obtain a family of genus 3 curves where every fiber is smooth.

Exercise 1.4. Consider a general pencil of quartic curves in the plane specializing to a quartic with a single node. Show that it is not possible to find a flat family of curves (even after base change) that replaces the central fiber with a smooth curve. (Hint: After blowing up the base points of the pencil, we can assume that the total space of the family is smooth and the surface is relatively minimal. First, assume we can replace the central fiber by a smooth curve without a base change. Use Zariski's main theorem to show that this is impossible. Then analyze what happens when we perform a base change.)

The previous exercise shows that the coarse moduli scheme of smooth curves (assuming it exists) cannot be proper. Given that curves in projective space can become arbitrarily singular, it is an amazing fact that the moduli space of curves can be compactified by allowing curves that have only nodes as singularities.

Definition 1.5. Consider the tuples $(C, p_1, \dots, p_n)$ where C is a connected at worst nodal curve of arithmetic genus g and $p_1, \dots, p_n$ are distinct smooth points of C. We call the tuple $(C, p_1, \dots, p_n)$ stable if in the normalization of the curve any rational component has at least three distinguished points—inverse images of nodes or of p_i—and any component of genus one has at least one distinguished point.

Note that for there to be any stable curves the inequality $2g - 2 + n > 0$ needs to be satisfied.

Definition 1.6. Let S be a scheme. A stable curve over S is a proper, flat family $C \to S$ whose geometric fibers are stable curves.

Theorem 1.7 (Deligne-Mumford-Knudsen)**.** *There exists a coarse moduli space $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed, genus g curves. $\overline{\mathcal{M}}_{g,n}$ is a projective variety and contains the coarse moduli space $\mathcal{M}_{g,n}$ of smooth n-pointed genus g curves as a Zariski open subset.*

One way to construct the coarse moduli scheme of stable curves is to consider pluri-canonically embedded curves, that is curves embedded in projective space $\mathbb{P}^{(2n-1)(g-1)-1}$ by their complete linear system $|nK_C|$ for $n \geq 3$. A locally closed subscheme K of the Hilbert scheme parameterizes the locus of n-canonical curves of genus g. The group $PGL((2n-1)(g-1))$ acts on K. The coarse moduli scheme may be constructed as the G.I.T. quotient of K under this action. The proof that this construction works is lengthy. Below we will briefly explain some of the main ingredients. We begin by recalling the key features of the construction of the Hilbert scheme. We then recall the basics of G.I.T..

2. A few remarks about the construction of the Hilbert scheme

Assume in this section that all schemes are Noetherian. Recall that the Hilbert functor is a contravariant functor from schemes to sets defined as follows:

Definition 2.1. Let $X \to S$ be a projective scheme, $\mathcal{O}(1)$ a relatively ample line bundle and P a fixed polynomial. Let

$$Hilb_P(X/S) : \{\text{Schemes}/S\} \to \{\text{sets}\}$$

be the contravariant functor that associates to an S scheme Y the subschemes of $X \times_S Y$ which are proper and flat over Y and have the Hilbert polynomial P.

A major theorem of Grothendieck asserts that the Hilbert functor is representable by a projective scheme.

Theorem 2.2. *Let X/S be a projective scheme, $\mathcal{O}(1)$ a relatively ample line bundle and P a fixed polynomial. The functor $Hilb_P(X/S)$ is represented by a morphism*

$$u : U_P(X/S) \to Hilb_P(X/S).$$

$Hilb_P(X/S)$ is projective over S.

I will explain some of the ingredients that go into the proof of this theorem, leaving you to read [Gr], [Mum2], [K], [Se] and the references contained in those accounts for complete details.

Let us first concentrate on the case $X = \mathbb{P}^n$ and $S = Spec(k)$, the spectrum of a field k. A subscheme of projective space is determined by its equations. The polynomials in $k[x_0, \ldots, x_n]$ that vanish on a subscheme form an infinite-dimensional subvector space of $k[x_0, \ldots, x_n]$. Suppose we knew that a finite-dimensional subspace actually determined the schemes with a fixed Hilbert polynomial. Then we would get an injection of the schemes with a fixed Hilbert polynomial into a Grassmannian. We have already seen that the Grassmannian (together with its tautological bundle) represents the functor classifying subspaces of a vector space. Assuming the image in the Grassmannian is an algebraic subscheme, we can use this subscheme to represent the Hilbert functor.

Given a proper subscheme Y of $\mathbb{P}^n$ and a coherent sheaf F on Y, the higher cohomology $H^i(Y, F(m))$, $i > 0$, vanishes for m sufficiently large. The finiteness that we are looking for comes from the fact that if we restrict ourselves to ideal sheaves of subschemes with a fixed Hilbert polynomial, one can find an integer m depending only on the Hilbert polynomial (and not on the subscheme) that works simultaneously for the ideal sheaf of every subscheme with a fixed Hilbert polynomial.

Theorem 2.3. *For every polynomial P, there exists an integer m_P depending only on P such that for every subsheaf $I \subset \mathcal{O}_{\mathbb{P}^n}$ with Hilbert polynomial P and every integer $k > m_P$*

(1) $h^i(\mathbb{P}^n, I(k)) = 0$ *for* $i > 0$*;*
(2) $I(k)$ *is generated by global sections;*
(3) $H^0(\mathbb{P}^n, I(k)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(\mathbb{P}^n, I(k+1))$ *is surjective.*

How does this theorem help? Let $Y \subset \mathbb{P}^n$ be a closed subscheme with Hilbert polynomial P. Choose $k > m_P$. By item (2) of the theorem, $I_Y(k)$ is generated by global sections. Consider the exact sequence

$$0 \to I_Y(k) \to \mathcal{O}_{\mathbb{P}^n}(k) \to \mathcal{O}_Y(k) \to 0.$$

This realizes $H^0(\mathbb{P}^n, I_Y(k))$ as a subspace of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$. This subspace determines $I_Y(k)$ and hence the subscheme Y. Since k depends only on the Hilbert polynomial, we get an injection to $G(P(k), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$. The image has a natural scheme structure. This scheme together with the restriction of the tautological bundle to it, represents the Hilbert functor. I will now fill in some of the details, leaving most of them to you. Let us begin with a sketch of the proof of the theorem.

Definition 2.4. A coherent sheaf F on $\mathbb{P}^n$ is called (Castelnuovo-Mumford) m-regular if $H^i(\mathbb{P}^n, F(m-i)) = 0$ for all $i > 0$.

Proposition 2.5. *If F is an m-regular coherent sheaf on $\mathbb{P}^n$, then*

(1) $h^i(\mathbb{P}^n, F(k)) = 0$ *for* $i > 0$ *and* $k + i \geq m$.
(2) $F(k)$ *is generated by global sections if* $k \geq m$.
(3) $H^0(\mathbb{P}^n, F(k)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(\mathbb{P}^n, F(k+1))$ *is surjective if* $k \geq m$.

Proof. The proposition is proved by induction on the dimension n. When $n = 0$, the result is clear. Take a general hyperplane H and consider the following exact sequence

$$0 \to F(k-1) \to F(k) \to F_H(k) \to 0.$$

When $k = m - i$, the associated long exact sequence of cohomology gives that

$$H^i(F(m-i)) \to H^i(F_H(m-i)) \to H^{i+1}(F(m-i-1)).$$

In particular, if F is m-regular on $\mathbb{P}^n$, then so is F_H on $\mathbb{P}^{n-1}$. Now we can prove the first item by induction on k. Now consider the similar long exact sequence

$$H^{i+1}(F(m-i-1) \to H^{i+1}(F(m-i)) \to H^{i+1}(F_H(m-i-1)).$$

The first group vanishes by induction on dimension and the third one vanishes by the assumption that F is m regular for $i \geq 0$. We conclude that F is $m+1$ regular. Hence by induction k regular for all $k > m$. This proves item (1).

Consider the commutative diagram

$$\begin{tikzcd}
 & H^0(F(k-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \arrow[r, "u"'] \arrow[d, "g"'] & H^0(F_H(k-1)) \otimes H^0(\mathcal{O}_H(1)) \arrow[d, "f"'] \\
H^0(F(k-1)) \arrow[r] & H^0(F(k)) \arrow[r, "v"'] & H^0(F_H(k))
\end{tikzcd}$$

The map u is surjective by the regularity assumption. The map f is surjective by induction on the dimension. It follows that $v \circ g$ is also surjective. Since the image of $H^0(F(k-1))$ is contained in the image of g, claim (3) follows.

It is easy to deduce (2) from (3). $\square$

The proof of the theorem is concluded if we can show that the ideal sheaves of proper subchemes of $\mathbb{P}^n$ with a fixed Hilbert polynomial are m_P-regular for an integer depending only on P. This claim also follows by induction on the dimension n. Choose a general hyperplane H and consider the exact sequence

$$0 \to I(m) \to I(m+1) \to I_H(m+1) \to 0.$$

I_H is a sheaf of ideals so we may use induction on the dimension.

Assume the Hilbert polynomial is given by

$$P(m) = \sum_{i=0}^{n} a_i \binom{m}{i}.$$

We then have

$$\chi(I_H(m+1)) = \chi(I(m+1)) - \chi(I(m))$$
$$= \sum_{i=0}^{n} a_i \left(\binom{m+1}{i} - \binom{m}{i} \right) = \sum_{i=0}^{n-1} a_{i+1} \binom{m}{i}$$

Assuming the result by induction, we get an integer m_1 depending only on the coefficients $a_1, \ldots, a_n$ such that I_H has that regularity. Considering the long exact sequence associated to our short exact sequence, we see that $H^i(I(m))$ is isomorphic to $H^i(I(m+1)$ as long as $i > 1$ and $m > m_1 - i$. Since by Serre's theorem these cohomologies vanish when m is large enough, we get the vanishing of the higher cohomology groups. For $i = 1$ we only get that $h^1(I(m))$ is strictly decreasing for $m \geq m_1 - 1$. We conclude that I is $m_1 + h^1(I(m_1 - 1))$-regular. However, since I is an ideal sheaf we can bound the latter term as follows

$$h^1(I(m_1-1)) = h^0(I(m_1-1)) - \chi(I(m_1-1)) \leq h^0(\mathcal{O}_{\mathbb{P}^n}(m_1-1)) - \chi(I(m_1-1)).$$

This clearly depends only on the Hilbert polynomial; hence concludes the proof of Theorem 2.3.

Now we indicate how one proceeds to deduce Theorem 2.2. So far we have given an injection from the set of subshemes of $\mathbb{P}^n$ with a fixed Hilbert polynomial P to the Grassmannian $G(P(m), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)))$ for any $m > m_P$ by sending the subscheme to the $P(m)$-dimensional subspace $H^0(\mathbb{P}^n, I(m))$ of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)))$. Of course, this subspace uniquely determines the subscheme. We still have to show that the image has a natural scheme structure and that this subscheme represents the Hilbert functor. For this purpose we will use flattening stratifications.

Recall that a stratification of a scheme S is a finite collection $S_1, \ldots, S_j$ of locally closed subschemes of S such that

$$S = S_1 \sqcup \cdots \sqcup S_j$$

is a disjoint union of these subschemes.

Proposition 2.6. *Let F be a coherent sheaf on $\mathbb{P}^n \times S$. Let S and T be Noetherian schemes. There exists a stratification of S such that for all morphisms $f : T \to S$, $(1 \times f)^* F$ to $\mathbb{P}^n \times T$ is flat over T if and only if the morphism factors through the stratification.*

This stratification is called the flattening stratification (see Lecture 8 in [Mum2] for the details). To prove it one uses the fact that if $f : X \to S$ is a morphism of finite type, S is integral and F is any coherent sheaf on X, then there is a dense open subset U of S such that the restriction of F to $f^{-1}(U)$ is flat over U. A corollary is that S can be partitioned into finitely many locally closed subsets S_i such that giving each the reduced induced structure, the restriction of F to $X \times_S S_i$ is flat over S_i.

We can partition S to locally closed subschemes as in the previous paragraph. Only finitely many Hilbert polynomials $P_i$occur. We can conclude that there is an integer m such that if $l \geq m$, then

$$H^i(\mathbb{P}^n(s), F(s)(l)) = 0$$

and

$$\pi_{S*} F(l) \otimes k(s) \to H^0(\mathbb{P}^n(s), F(s)(l))$$

is an isomorphism, where π_S denotes the natural projection to S.

Next one observes that $(1 \times f)^* F$ is flat over T if and only if $f^* \pi_{S*} F(l)$ is locally free for all $l \geq m$. For each l we find the stratification of S such that $S_{l,j}$ the sheaf $f^* \pi_{S*} F(l)$ is locally free of rank j. Note that there is the following equality between subsets of S

$$\cap_{l \geq m} \mathrm{Supp}[S_{l,j}] = \cap_{m+n \geq l \geq m} \mathrm{Supp}[S_{l,j}].$$

This is because the Hilbert polynomials have degree at most n.

For each integer $h \geq 0$, there is a well-defined locally closed subscheme of S defined by

$$\cap_{0 \leq r \leq h} S_{r, P_i(m+r)}.$$

When $h \geq n$, these form a decreasing sequence of subschemes with the same support. Therefore, they stabilize. These give us the required stratification.

The flattening stratification allows us to put a scheme structure on the image of our map to the Grassmannian. More precisely, consider the incidence correspondence

$$I \subset \mathbb{P}^n \times G(P(m_P), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m_P))).$$

The incidence correspondence has two projections

$$\pi_1 : I \to \mathbb{P}^n$$

and

$$\pi_2 : I \to G(P(m_P), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m_P))).$$

For the rest of this section we will abbreviate $G(P(m_P), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m_P)))$ simply by G. $\pi_2^* T(-m_P)$ where T is the tautological bundle on G is an idea sheaf of $\mathcal{O}_{\mathbb{P}^n \times G}$. Let us denote the corresponding subscheme by Y. The flattening stratification of $\mathcal{O}_Y$ over G gives a subscheme H_P of G corresponding to the Hilbert polynomial P. (Note that this is the scheme structure that we put on the set we earlier obtained.) The claim is that H_P represents the Hilbert functor and the universal family is the restriction W of Y to the inverse image of H_P.

Suppose we have a subscheme $X \subset \mathbb{P}^n \times S$ mapping to S via f and flat over S (and suppose the Hilbert polynomial is P). We obtain an exact sequence

$$0 \to f_* I_X(m_P) \to f_* \mathcal{O}_{\mathbb{P}^n \times S}(m_P) \to f_* \mathcal{O}_X(m_P) \to 0.$$

By the universal property of the Grassmannian G, this induces a map $g : S \to G$. Since

$$f_* I_X(m) = g^* \pi_{2*} I_Y(m)$$

for m sufficiently large, we see that $(1 \times g)^* \mathcal{O}_Y$ is flat with Hilbert polynomial P, hence g factors through H_P by the definition of the flattening stratification. Moreover, X is simply $S \times_{H_P} W$. This concludes the construction of $Hilb_P(\mathbb{P}^n/S)$.

Exercise 2.7. Verify the details of the above construction.

So far we have constructed the Hilbert scheme as a quasi-projective subscheme of the Grassmannian. To prove that it is projective it suffices to check that it is proper. This is done by checking the valuative criterion of properness. This follows from the following proposition [Ha] III.9.8.

Proposition 2.8. *Let X be a regular, integral scheme of dimension one. Let $p \in X$ be a closed point. Let $Z \subset \mathbb{P}^n_{X-p}$ be a closed subscheme flat over $X - p$. Then there exists a unique closed subscheme $\overline{Z} \in \mathbb{P}^n_X$ flat over X, whose restriction to $\mathbb{P}^n_{X-p}$ is Z.*

Exercise 2.9. Deduce from the proposition that the Hilbert scheme we constructed is projective.

Exercise 2.10. For a projective scheme X/S construct $Hilb_P(X/S)$ as a locally closed subscheme of $Hilb_P(\mathbb{P}^n/S)$.

Exercise 2.11. Suppose X and Y are projective schemes over S. Assume X is flat over S. Let $Hom(X, Y)$ be the functor that associates to any S scheme T the set of morphisms

$$X \times_S T \to Y \times_S T.$$

Using our construction of the Hilbert scheme and noting that a morphism may be identified with its graph construct a scheme that represents the functor $Hom(X, Y)$.

2.1. **Examples of Hilbert schemes.** In this subsection we would like to give some explicit examples of Hilbert schemes.

Example 2.12. Consider the Hilbert scheme associated to a projective variety X and the Hilbert polynomial 1. Then the Hilbert scheme is simply X.

Exercise 2.13. Show that if C is a smooth curve, then $Hilb_n(C)$ is simply the symmetric n-th power of C. In particular, $Hilb_n(\mathbb{P}^1) = \mathbb{P}^n$

Exercise 2.14. Show that the Hilbert scheme of hypersurfaces of degree d in $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^{\binom{n+d}{d}-1}$.

Example 2.15 (The Hilbert scheme of conics in $\mathbb{P}^3$)**.** Any degree 2 curve is necessarily the complete intersection of a linear and quadratic polynomial. Moreover, the linear polynomial is uniquely determined. We thus obtain a map

$$Hilb_{2n-1}(\mathbb{P}^3) \to \mathbb{P}^{3*}.$$

The fibers of this map are $Hilb_{2n-1}(\mathbb{P}^2)$ which is isomorphic to $\mathbb{P}^5$. We conclude by Zariski's main theorem that that $Hilb_{2n-1}(\mathbb{P}^3)$ is the $\mathbb{P}^5$ bundle $\mathbb{P}(Sym^2 T^*) \to \mathbb{P}^{3*}$. Of course, in all this discussion we needed the fact that $Hilb_{2n-1}(\mathbb{P}^3)$ is reduced.

Theorem 2.16. *Let X be a projective scheme over a field k and $Y \subset X$ be a closed subscheme, then the Zariski tangent space to $Hilb(X)$ at $[Y]$ is naturally isomorphic to $Hom_Y(I_Y/I_Y^2, \mathcal{O}_Y)$.*

In particular, in our case the dimension of $THilb_{2n-1}(\mathbb{P}^3) = h^0(N_{C/\mathbb{P}^3}) = 8$. Hence $Hilb_{2n-1}(\mathbb{P}^3)$ is reduced (in fact smooth). $Hilb_{2n-1}(\mathbb{P}^3)$ is one of the few examples where we can answer many of the geometric questions we can ask about a Hilbert scheme.

We can use the Hilbert scheme of conics to solve the following question:

Question 2.17. How many conics in $\mathbb{P}^3$ intersect 8 general lines in $\mathbb{P}^3$?

As in the case of Schubert calculus, we can try to calculate this number as an intersection in the cohomology ring. The cohomology ring of a projective bundle over a smooth variety is easy to describe in terms of the chern classes of the bundle and the cohomology ring of the variety.

Theorem 2.18. *Let E be a rank n vector bundle over a smooth, projective variety X. Suppose that the chern polynomial of E is given by $\sum c_i(E)t^i$. Let ζ denote the first chern class of the dual of the tautological bundle over $\mathbb{P}E$. The cohomology of $\mathbb{P}E$ is isomorphic to*

$$H^*(\mathbb{P}E) \cong \frac{H^*(X)\ [\zeta]}{< \zeta^n + \zeta^{n-1}c_1(E) + \cdots + c_n(E) = 0 >}$$

If you are not familiar with chern classes, see the handout about chern classes. Using Theorem 2.18 we can compute the cohomology ring of $Hilb_{2n-1}(\mathbb{P}^3)$. Recall that T^* on $\mathbb{P}^{3*}$ is a rank 3 vector bundle with chern polynomial

$$c(T^*) = 1 + h + h^2 + h^3.$$

Using the splitting principle we assume that the polynomial splits into three linear factors

$$(1 + x)(1 + y)(1 + z).$$

Then the chern polynomial of $Sym^2(T^*)$ is given by

$$(1 + 2x)(1 + 2y)(1 + 2z)(1 + x + y)(1 + x + z)(1 + y + z).$$

Multiplying this out and expressing it interms of the elementary symmetric polynomials in x, y, z, we see that

$$c(Sym^2(T^*)) = 1 + 4h + 10h^2 + 20h^3.$$

It follows that the cohomology ring of $Hilb_{2n-1}(\mathbb{P}^3)$ is given as follows:

$$H^*(Hilb_{2n-1}(\mathbb{P}^3)) \cong \frac{\mathbb{Z}[h, \zeta]}{< h^4,\ \zeta^3 + 4h\zeta^2 + 10h^2\zeta + 20h^3 >}$$

The class of the locus of conics interseting a line is given by $2h + \zeta$. This can be checked by a calculation away from codimension at least 2. Consider the locus of planes in $\mathbb{P}^{3*}$ that do not contain the line l. Over this locus there is a line bundle that associates to each point (H, Q) on $Hilb_{2n-1}(\mathbb{P}^3)$ the homogeneous quadratic polynomials modulo those that vanish at $H \cap l$. This line bundle is none other than the pull-back of $\mathcal{O}_{\mathbb{P}^{3*}}$. The tautological bundle over $Hilb_{2n-1}(\mathbb{P}^3)$ maps by evaluation. The locus where the evaluation vanishes is the locus of conics that intersect l. Hence the class is the difference of the first chern classes. Finally, we compute $(2h + \zeta)^8$ using the presentation of the ring to obtain 92.

Over the complex numbers we can invoke Kleiman's theorem to deduce that there are 92 smooth conics intersecting 8 general lines in $\mathbb{P}^3$.

Exercise 2.19. Calculate the number of conics that intersect $8 - 2i$ lines and contain i points for $0 \leq i \leq 3$.

Exercise 2.20. Calculate the class of conics that are tangent to a plane in $\mathbb{P}^3$. Find how many conics are tangent to a general plane and intersect 7 general lines.

Exercise 2.21. Generalize the previous discussion to conics in $\mathbb{P}^4$. Calculate the numbers of conics that intersect general $11 - 2i - 3j$ planes, i lines and j points.

Example 2.22 (The Hilbert scheme of twisted cubics in $\mathbb{P}^3$)**.** The Hilbert polynomial of a twisted cubic is $3t + 1$. This Hilbert scheme has two components. A general point of the first component parameterizes a smooth rational curve of degree 3 in $\mathbb{P}^3$. A general point of the second component parameterizes a degree

3 plane curve together with a point in $\mathbb{P}^3$. Note that the dimension of the first component is 12, whereas the dimension of the second component is 15. Hence the Hilbert scheme is not pure dimensional. The component of the Hilbert scheme parameterizing the smooth rational curves has been studies in detail. In fact, that component is smooth.

Exercise 2.23. Describe the subschemes of $\mathbb{P}^3$ that are parameterized by the component of the Hilbert scheme that parameterizes smooth rational curves of degree 3 in $\mathbb{P}^3$.

Piene and Schlessinger proved that the component of the Hilbert scheme parameterizing twisted cubics is smooth. In analogy with our analysis of the Hilbert scheme of conics we can try to compute invariants of cubics using the Hilbert scheme. Unfortunately, this turns out to be very difficult.

Problem 2.24. Calculate the number of twisted cubics intersecting 12 general lines in $\mathbb{P}^3$.

Problem 2.25. Calculate the number of twisted cubics that are tangent to 12 general quadric hypersurfaces in $\mathbb{P}^3$. (Hint: There are 5,819,539,783,680 of them.)

Towards the end of the course we will see how to use the Kontsevich moduli space to answer these questions.

Unfortunately, Hilbert schemes are often unwieldy schemes to work with. They often have many irreducible components. It is hard to compute the dimensions of these components. Even components of the Hilbert scheme whose generic point parameterizes smooth curves in $\mathbb{P}^3$ may be everywhere non-reduced.

Example 2.26 (Mumford's example)**.** Mumford showed that there exists a component of the Hilbert scheme parameterizing smooth curves of degree 14 and genus 24 in $\mathbb{P}^3$ that is non-reduced at the generic point of that component. See [Mum1] or [HM] Chapter 1 Section D.

The pathological behavior of most Hilbert schemes make them hard to use for studying the explicit geometry of algebraic varieties. In fact, the Hilbert schemes often exhibit behavior that is arbitrarily bad. For instance, R. Vakil recently proved that all possible singularities occur in some component of the Hilbert scheme of curves in projective space.

Theorem 2.27 (Murphy's Law)**.** *Every singularity class of finite type over Spec$\mathbb{Z}$ occurs in a Hilbert scheme of curves in some projective space.*

3. Basics about curves

Here we collect some basic facts about stable curves.

If $\pi : C \to S$ is a stable curve of genus g over a scheme S, then C has a relative dualizing sheaf $\omega_{C/S}$ with the following properties

1. The formation of $\omega_{C/S}$ commutes with base change.
2. If $S = Spec\ k$ where k is an algebraically closed field and $\tilde{C}$ is the normalization of C, then $\omega_{C/S}$ may be identified with the sheaf of meromorphic differentials on $\tilde{C}$ that are allowed to have simple poles only at the inverse image of the nodes subject to the condition that if the points x and y lie over the same node then the residues at these two points must sum to zero.

(3) In particular, if C is a stable curve over a field k, then $H^1(C, \omega_{C/k}^{\otimes n}) = 0$ if $n \geq 2$ and $\omega_{C/k}^{\otimes n}$ is very ample for $n \geq 3$. When $n = 3$ we obtain a tri-canonical embedding of stable curves to $\mathbb{P}^{5g-6}$ with Hilbert polynomial $P(m) = (6m-1)(g-1)$.

To see the third property observe that every irreducible component E of a stable curve C either has arithmetic genus 2 or more, or has arithmetic genus one but meets the other components in at least one point, or has arithmetic genus 0 and meets the other components in at least three points. Since $\omega_{C/k} \otimes \mathcal{O}_E$ is isomorphic to $\omega_{E/k}(\sum_i Q_i)$ where Q_i are the points where E meets the rest of the curve. Since this sheaf has positive degree it is ample on each component E of C, hence it is ample. $\omega_{E/k}(\sum_i Q_i)$ has positive degree on each component, hence $\omega_{C/k}^{1-n} \otimes \mathcal{O}_E$ has no sections for any $n \geq 2$. By Serre duality, it follows that $H^1(C, \omega_{C/k}^{\otimes n}) = 0$. To show that when $n \geq 3$, $\omega_{C/k}^{\otimes n}$ is very ample, it suffices to check that $\omega_{C/k}^{\otimes n}$ separates points and tangents.

Exercise 3.1. Check that when $n \geq 3$, $\omega_{C/k}^{\otimes n}$ separates points and tangents.

4. Stable reduction

Stable reduction was originally proved by Deligne and Mumford using the existence of stable reduction for abelian varieties [DM]. [HM] Chapter 3 Section C contains a beautiful account which we will summarize below.

The main theorem is the following:

Theorem 4.1 (Stable reduction)**.** *Let B be the spectrum of a DVR with function field K. Let $X \to B$ be a family of curves with n sections $\sigma_1, \dots, \sigma_n$ such that the restriction $X_K \to$ Spec K is an n-pointed stable curve. Then there exists a finite field extension L/K and a unique stable family $\tilde{X} \to B \times_K L$ with sections $\tilde{\sigma}_1, \dots, \tilde{\sigma}_n$ such that the restriction to Spec L is isomorphic to $X_K \times_K L$.*

One can algorithmically carry out stable reduction (at least in characteristic zero). Since stable reduction is an essential tool in algebraic geometry we begin by giving some examples. We will then sketch the proof.

Example 4.2. Fix a smooth curve C of genus $g \geq 2$. Let $p \in C$ be a fixed point and let q be a varying point. More precisely, we have the family $C \times C \to C$ with two sections $\sigma_p : C \to C \times C$ mapping a point q to (q, p) and $\sigma_q : C \to C \times C$ mapping q to (q, q). All the fibers are stable except when $p = q$. To obtain a stable family, we blow up $C \times C$ at (p, p). The resulting picture looks as follows (see Figure 2):

There is an algorithm that produces the stable reduction in characteristic zero. This algorithm is worth knowing because the explicit calculation of the stable limit often has applications to geometric problems.

Step 1. Resolve the singularities of the total space of the family. The result of this step is a smooth surface X mapping to our initial surface. Moreover, we can assume that the support of the central fiber is a normal-crossings divisor.

Step 2. After Step 1 at every point of the central fiber the pull-back of the uniformizer may be expressed as x^a for some $a > 0$ at a smooth point or $x^a y^b$ for

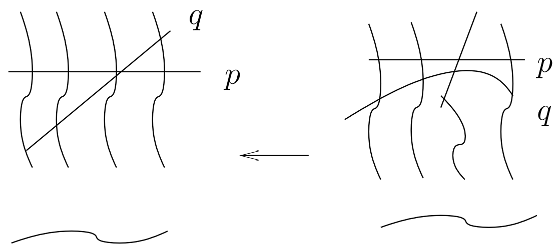

FIGURE 2. Stable reduction when two marked points collide.

a pair $a, b > 0$ at a node. Make a base change of order p for some prime dividing the multiplicity of a multiple component of the fiber.

Step 3. Normalize the resulting surface.

Suppose the central fiber was of the form $\sum_i n_i C_i$ The effect of doing steps 2 and 3 is to take a branched cover of the surface X branched along the reduction of the divisor forming the central fiber modulo p. Repeat steps 2 and 3 until all the components occuring in the central fiber appear with multiplicity 1.

Step 4. Contract the rational components of the central fiber that are not stable.

Sketch of proof of Theorem 4.1. We will assume that $n = 0$ and then make some remarks about how to modify the statements here to obtain the general case. Let R be a DVR with uniformizer z. Let $\eta \in B = $ *Spec* R be the generic point. We are assuming that our family X_η is a stable curve of genus g.

Consider regular, proper B-schemes that extend X_η. By results of Abhyankar [Ab] about resolutions of surface singularities there exists a unique relatively minimal model of X_η. Consider the completion of the local ring at a node of the special fiber. This ring is isomorphic to $R[[x, t]]/(xy - z^n)$ for some integer $n \geq 1$. This ring is not regular for $n > 1$. We can desingularize it in a sequence of $\lfloor n/2 \rfloor$ blow-ups. Over the node we get a sequence of -2-curves.

Let X be a proper, flat regular surface extending X_η. Let C_i, $i = 1, \ldots, n$, be the components of the special fiber. Suppose they occur with multiplicity r_i. Recall the following basic facts about the components of the special fiber

(1) The special fiber C is connected and the multiplicities $r_i > 0$ for all i.
(2) $C_i \cdot C_j \geq 0$ for all $i \neq j$ and $C_i \cdot C = 0$ for all i.
(3) If K is the canonical class, then the arithmetic genus of C_i is given by the genus formula as
$$1 + \frac{C_i^2 + C_i \cdot K}{2}.$$
(4) The intersection matrix $C_i \cdot C_j$ is a negative definite symmetric matrix. The only linear combinations $Z = \sum a_i C_i$ with the property that $Z^2 = 0$ are rational multiples of C.

One can divide the components C_i of the special fiber into the following categories

□

Example 4.3. Suppose we have a general pencil of smooth curves of genus g in $\mathbb{P}^2$ specializing to a curve with an ordinary m-fold point. We may write down the equation of such a family as $F + tG$ where G is the equation defining a general curve of genus g and F locally has the form

$$\prod_{i=1}^{m}(y - a_i x) \ + \ \text{h. o. t.}$$

with distinct a_i. To perform stable reduction we blow-up the m-fold point. In the resulting surface the proper transform C of the central fiber is smooth of genus $g - m(m-1)/2$, but the exceptional divisor is a $\mathbb{P}^1$ that meets C in m points and occurs with multiplicity m. We make a base change of order m. We get an m-fold cover of this $\mathbb{P}^1$ totally ramified at the m points of intersection with C. By the Riemann-Hurwitz formula this is a genus $m(m-3)/2+1$. The stable limit then is as shown in the figure.

Exercise 4.4. Suppose C_t is a general pencil of smooth genus g plane curves acquiring an ordinary cusp (a singularity whose local equation is given by $y^2 = x^3$). Describe the stable limit of this family of curves.

Exercise 4.5. Read and do the exercises in Chapter 3 Section C of [HM].

5. Deligne-Mumford Stacks

In this section for completeness I will give you the definition of Deligne-Mumford stacks. I will summarize a few basic results and definitions. Much better accounts exist in [DM], [Ed] and [LM-B]. See also [Fan].

Let $\mathcal{S}$ be the category of schemes over a scheme S. A category T over $\mathcal{S}$ is a category together with a functor $p : T \to \mathcal{S}$.

Definition 5.1 (Groupoid)**.** A category $((T, p)$ over $\mathcal{S}$ is a groupoid if the following two conditions hold

(1) If $f : B' \to B$ is a morphism in $\mathcal{S}$ and C is an object in T lying over B, then there exists an object C' over B' and a morphism $\phi : C' \to C$ such that $p(\phi) = f$.
(2) Let C, C', C'' be objects in T lying over the objects B, B', B'' in $\mathcal{S}$, respectively. If $\phi : C' \to C$ and $\psi : C'' \to C$ are morphisms in T and $f : B' \to B''$ is a morphism in $\mathcal{S}$ satisfying $p(\psi) \circ f = p(\phi)$, then there is a unique morphism $\tau : C' \to C''$ such that $\psi \circ \tau = \phi$ and $p(\tau) = f$.

Example 5.2. Recall that a Deligne-Mumford stable curve (or simply a stable curve) of genus $g \geq 2$ over a scheme S is a proper, flat family $\pi : C \to S$ whose geometric fibers are reduced, connected, one dimensional schemes C_s satisfying the following properties:

(1) The only singularities of C_s are ordinary double points.
(2) A non-singular rational component of C_s meets the other components in at least three points.
(3) C_s has arithmetic genus g—equivalently $h^1(\mathcal{O}_{C_s}) = g$.

We can define a groupoid $\overline{\mathcal{M}}_g$ of Deligne-Mumford stable curves of genus g over schemes over *Spec* $\mathbb{Z}$ as follows: The sections of $\overline{\mathcal{M}}_g$ over a scheme X are families of stable curves $C \to X$. A morphism between $C' \to X'$ and $C \to X$ is a fiber diagram

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

which induces an isomorphism $C' \cong X' \times_X C$.

$\overline{\mathcal{M}}_g$ is a groupoid and it is the main example that we are interested in.

For the sake of future constructions and definitions it is important to keep in mind the examples of two more groupoids.

Example 5.3. Any contravariant functor $F : \mathcal{S} \to \{\text{sets}\}$ from schemes to sets gives rise to a groupoid (usually also called F by abuse of notation). The objects of the groupoid F are pairs (X, α) where X is a scheme and α is an element of the set $F(X)$. A morphism between (X, α) and (Y, β) is a morphism $f : X \to Y$ such that $F(f)(\beta) = \alpha$. In particular, this construction allows us to view schemes as groupoids. To a scheme X we can associate its functor of points $\mathrm{Hom}(*, X)$. Since this is a contravariant functor from schemes to sets, to a scheme X we can also associate a groupoid X. The distinction between a scheme X and the associated groupoid is often blurred.

Example 5.4. Since the construction of many moduli spaces involves taking the quotient of a parameter space (such as a component of a Hilbert scheme) by a group action, the groupoid $[X/G]$ is important. Let X be a scheme and G a group scheme acting on X. The sections of $[X/G]$ over a scheme Y are principal G-bundles $E \to Y$ together with a G-equivariant map $E \to X$. A morphism between two such principal G-bundles is a pull-back diagram.

Exercise 5.5. There is a relation between the previous two examples. Show that if the action of G on X is free and a quotient scheme X/G exists, then then there is an equivalence of categories between $[X/G]$ and the groupoid associated to the scheme X/G.

Let (T, p) be a groupoid. For any two objects X and Y in the fiber of T over a scheme B, we can associate a functor $Isom_B(X, Y)$. This functor associates to any morphism $f : B' \to B$, the set of isomorphisms in $T(B')$ between $f^*(X)$ and $f^*(Y)$.

In the case of Deligne-Mumford stable curves, given any two stable curves C and C', $Isom_X(C, C')$ associates to any morphism $f : Y \to X$ the set of isomorphisms between $f^*(C)$ and $f^*(C')$. Recall that C and C' are both canonically polarized by $\omega_{C/X}$ and $\omega_{C'/X}$, respectively. Moreover, the formation of the relative dualizing sheaf commutes with base change. Consequently, any isomorphism satisfies $f^*(\omega_{C'/X}) = \omega_{C/X}$. Hence, all isomorphisms are isomorphisms between polarized schemes. It follows by the existence of the Hilbert scheme, that $Isom_X(C, C')$ is represented by a scheme quasi-projective over X.

Definition 5.6 (Stack)**.** A groupoid (T, p) over $\mathcal{S}$ is a stack if

(1) $Isom_B(X, Y)$ is a sheaf in the étale topology for all B, X and Y;

(2) If $\{B_i \to B\}$ is a covering of B in the étale topology, and X_i are a collection of objects in $T(B_i)$ with isomorphisms

$$\phi_{i,j} : X_{j|B_i \times_B B_j} \to X_{i|B_i \times_B B_j}$$

in $T(B_i \times_B B_j)$ satisfying the cocycle condition, then there exists an object $X \in T(B)$ with isomorphisms $X_{|B_i} \to X_i$ inducing the isomorphisms $\phi_{i,j}$.

Example 5.7. The groupoid $[X/G]$ defined in Example 5.4 is a stack. Let e, e' be two objects in $[X/G](Y)$ corresponding to two principal G-bundles $E, E' \to Y$ with G-equivariant maps f, f' to X, respectively. $Isom_Y(e, e')$ is empty unless $E = E'$ and $f = f'$. In the latter case the isomorphisms correspond to the subgroup of G that stabilizes the map f. Since the functor that associates to a G-equivariant map its stabilizer is representable, condition (1) follows. Condition (2) also holds for principal G-bundles.

Let $P_{g,n}(m)$ be the Hilbert polynomial $(2nm-1)(g-1)$, the Hilbert polynomial of an n-canonically embedded stable curve. Set $N = n(2g-2)-g$. Let $\overline{H}_{g,n}$ the subscheme of the Hilbert scheme $Hilb_{(2nm-1)(g-1)}(\mathbb{P}^N)$ parameterizing n-canonically embedded stable curves. Below we will show that there is an equivalence of categories between $\overline{\mathcal{M}}_g$ and $[\overline{H}_{g,n}/\mathbb{P}GL(N+1)]$ where the action of $\mathbb{P}GL(N+1)$ on the Hilbert scheme is the one induced by its usual action on $\mathbb{P}^N$. In particular, it follows from the previous example that $\overline{\mathcal{M}}_g$ is a stack.

Recall in example 5.3 we associated to a scheme a groupoid. Observe that this groupoid is a stack. The second condition is satisfied because the functor of points of a scheme is represented by the scheme itself. In particular, we can view each scheme as a stack. In the litterature stacks that arise this way are usually referred to as schemes meaning that the stack associated to the scheme. We will also indulge in this habit.

A morphism of stacks $f : T \to T'$ is representable if for any map of a scheme $X \to T'$ the fiber product $T \times_{T'} X$ is represented by a scheme. We can transport the notions of morphisms of schemes to representable morphisms of stacks in the following way: We say that a representable morphism $f : T \to T'$ has a property P (such as quasi-compact, separated, proper, etc.) if for all maps of a scheme $X \to T'$, the corresponding morphism of schemes $T \times_{T'} X \to X$ has the property P.

Definition 5.8 (Deligne-Mumford stack)**.** A stack is called a Deligne-Mumford stack if

(1) The diagonal $\Delta_X : T \to T \times_{\mathcal{S}} T$ is representable, quasi-compact and separated;
(2) There exists a scheme U and an étale, surjective morphism $U \to T$.

Morphisms as in condition (2) are called étale atlases.

The following is a useful theorem for verifying that a stack is a Deligne-Mumford stack (see [DM] Theorem 4.21, or [Ed] Theorem 2.1).

Theorem 5.9. *Let T be a quasi-separated stack over a Noetherian scheme S. Suppose that*

(1) *The diagonal is representable and unramified,*
(2) *There exists a scheme U of finite type over S and a smooth, surjective S-morphism $U \to F$.*

The F is a Deligne-Mumford stack.

A consequence of this theorem is that if X/S is a Noetherian scheme of finite type and G/S is a smooth group scheme acting on X with with finite and reduced stabilizers, then $[X/G]$ is a Deligne-Mumford stack. The conditions on the stabilizers (that they are finite and reduced) guarantee that $Isom_B(E, E)$ are unramified. It follows that the diagonal is unramified. The second condition in the theorem is satisfied by the map $X \to [X/G]$.

Given the equivalence of categories between $\overline{\mathcal{M}}_g$ and $[\overline{H}_{g,n}/\mathbb{P}GL(N+1)]$ it follows that $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack because the action of $\mathbb{P}GL(N+1)$ on $\overline{H}_{g,n}$ has finite and reduced stabilizers.

Just like in the case of schemes there are valuative criteria for separatedness and properness. We now state these and observe that $\overline{\mathcal{M}}_g$ is a proper Deligne-Mumford stack. For the following two theorems let $f : T \to S$ be a morphism of finite type from a Deligne-Mumford stack to a noetherian scheme S

Theorem 5.10 (The valuative criterion for separatedness)**.** *The morphism f is separated if and only if for any complete discrete valuation ring with algebraically closed residue field and any commutative diagram*

$$\begin{tikzcd} & T \arrow[d, "f"] \\ Spec\ R \arrow[ur, "g_1,g_2"] \arrow[r] & S \end{tikzcd}$$

any isomorphism between the restrictions of g_1 and g_2 to the generic point of Spec R can be extended to an isomorphism of g_1 and g_2.

Theorem 5.11 (The valuative criterion of properness)**.** *If f is separated, then f is proper if and only if, for any discrete valuation ring R with field of fractions K and any map Spec $R \to T$ which lifts over Spec K to a map to T, there is a finite extension K' of K such that the lift extends to all of Spec R' where R' is the integral closure of R in K'.*

The stable reduction theorem together with the valuative criterion of properness implies that $\overline{\mathcal{M}}_g$ is a proper Deligne-Mumford stack.

One approach for constructing the coarse moduli scheme (which we cannot complete at present because we have not yet developed the theory of divisors on the moduli stack) is to first construct the moduli space as an algebraic space, then exhibit an ample divisor on the coarse moduli algebraic space. This approach has been applied successfully to represent many moduli functors. The first step is achieved by a corollary of a general theorem of Keel and Mori [KM] (see also [Li] for a nice treatment).

Theorem 5.12. *Any separated Deligne-Mumford stack of finite type has a coarse moduli space in the category of algebraic spaces.*

Once we study the ample cone in the Picard group of the moduli stack, we will be able to deduce the existence of a coarse moduli scheme from the previous theorem. The second approach to the construction of the coarse moduli scheme is to directly take the G.I.T. quotient of the Hilbert scheme parameterizing n-canonically embedded stable curves. The advantage of the first approach is that it does away with delicate calculations describing the stable and semi-stable loci of this action. The first approach may also be used to construct moduli spaces in other situations. The advantage of the second approach is that it produces a projective coarse moduli scheme at once.

6. The GIT construction of the moduli space

Good references for this section are [HM] Chapter 4, [Mum3], [FKM] and [Ne]. Explaining the GIT construction in detail would take us too far afield. Instead we will briefly sketch the main ideas and refer you to the literature.

6.1. **Basics about G.I.T..**

An algebraic group G is a group together with the structure of an algebraic variety such that the multiplication and inverse maps are morphisms of varieties. An action of an algebraic group G on a variety X is a morphism $f : G \times X \to X$ such that $f(gg', x) = f(g, f(g', x))$ and $f(e, x) = x$, where e is the identity of the group. The stabilizer of a point $x \in X$ is the closed subgroup of G fixing x. The orbit of a point x under G is the image of f restricted to $G \times \{x\}$.

For our purposes we can always restrict attention to $SL(n)$, $GL(n)$ or $\mathbb{P}GL(n)$. An algebraic group which is isomorphic to a closed subgroup of $GL(n)$ is called a linear algebraic group. A group is called geometrically reductive if for every linear action of G on k^n and every non-zero invariant point $v \in k^n$, there exists an invariant homogeneous polynomial that does not vanish on v. The group is called linearly reductive if the homogeneous polynomial may be taken to have degree one. Finally a group is called reductive if the maximal connected normal solvable subgroup is isomorphic to a direct product of copies of k^*. In characteristic zero these concepts coincide. In characteristic $p > 0$ a threorem of Haboush guarantees that every reductive group is geometrically reductive.

The question is to obtain a quotient of a variety under the action of a reductive group.

Lemma 6.1. *Let G be a geometrically reductive group acting on an affine variety X. Let W_1 and W_2 be two disjoint invariant closed orbits. Then there exists an invariant polynomial $f \in A(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$.*

Proof. Pick any $h \in A(X)$ such that $h(W_1) = 0$ and $h(W_2) = 1$. Consider the subspace spanned by h^g for $g \in G$. This is a finite dimensional subspace. To see this consider the function $H(g, x) = h(gx)$ in $A(G \times X) \cong A(G) \otimes A(X)$. We can write $H(g, x)$ as a finite sum $\sum_i F_i \otimes H_i$ in $A(G) \otimes A(X)$ of the generators of $A(G)$ and $A(X)$. Hence the subspace spanned by h^g for $g \in G$ is contained in the subspace spanned by the H_i. Pick a basis for this subspace $h_1, \dots, h_n$. We obtain a rational representation of G on this subspace, hence a linear action on k^n making the morphism $\pi : X \to k^n$ given by $\pi(x) = (h_1(x), \dots, h_n(x))$ into a G-morphism. Since G is geometrically reductive there is an invariant polynomial f that has the value zero on $\pi(W_1)$ and the value 1 on $\pi(W_2)$. $f \circ \pi$ is the desired polynomial. $\square$

The main theorem for quotients of reductive group actions on affine varieties is the following:

Theorem 6.2. *Let G be a reductive group acting on an affine variety X. Then there exists a quotient affine variety Y and a G-invariant, surjective morphism $\phi : X \to Y$ such that*

(1) *For any open set $U \subset Y$, the ring homomorphism*
$$\phi^* : A(U) \to A(\phi^{-1}(U))$$
is an isomorphism of $A(U)$ with $A(\phi^{-1}(U))^G$.

(2) *If $W \subset X$ is a closed invariant subset, then $\phi(W)$ is closed in Y.*

(3) *If W_1 and W_2 are disjoint closed invariant sets, then their images under ϕ are disjoint.*

Proof. The main technical results are provided by a theorem of Haboush and a theorem of Nagata.

Theorem 6.3 (Haboush)**.** *Any reductive group G is geometrically reductive.*

Theorem 6.4 (Nagata)**.** *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra R. Then the ring of invariants R^G is finitely generated.*

In view of these theorems $A(X)^G$ is finitely generated. Hence we can let $Y = Spec\ A(X)^G$. The inclusion of $A(X)^G \to A(X)$ induces a morphism $\phi : X \to Y$. The claimed properties are easy to check for ϕ. □

Remark 6.5. The following are straightforward observations:

(1) For any open subset $U \subset Y$, (U, ϕ) is a categorical quotient of $\phi^{-1}(U)$ by G.

(2) The images of two points in X coincide if and only if the orbit closures of these two points intersect. Consequently, Y will be an orbit space if and only if the orbits of the G action on X are closed.

Remark 6.6. We will not prove Haboush's theorem here. The interested reader may consult the original paper [Hab]. Over the complex numbers reductive, geometrically reductive and linearly reductive coincide. This follows from the fact that any finite dimensional representation is decomposible to irreducible representations. Projection to the one-dimensional invariant subspace produces the desired invariant linear functional.

We now sketch the proof of Nagata's theorem. Since R is a finitely generated k-algebra, we can pick generators $f_1, dots, f_n$ that generate R. We can also assume that the subspace spanned by the f_i is G-invariant. (If not, we can replace it by a minimal G-invariant subspace, which is finite-dimensional by the argument in Lemma 6.1.) We thus obtain a linear G action on the subspace spanned by f_i by setting
$$f_i^g = \sum_j \alpha_{i,j}(g) f_j.$$
Let $S = k[X_1, \ldots, X_n]$. There is an action of G on S by setting
$$X_i^g = \sum_j \alpha_{i,j}(g) X_j.$$
There is a k-algebra homomorphism from S to R sending X_i to f_i that is compatible with the G actions. We are thus reduced to proving Nagata's theorem in the case

when G acts on S preserving degree, $Q \subset S$ is a G-invariant ideal with the induced action on $R = S/Q$. Under these assumptions we would like to see R^G is finitely generated.

Suppose not. Since S is Noetherian, there exists an ideal Q maximal among those that are G-invariant such that R^G where $R = S/Q$ is not finitely generated. Then if $J \neq 0$ is a G-invariant homogeneous ideal in R, then $(R/J)^G$ is finitely generated. Suppose first there is a homogeneous ideal Q with the desired properties.

I claim that $(R/J)^G$ is integral over $R^G/(J \cap R^G)$. Suppose $f \in (R/J)^G$. Pick $h \in R$ such that the image of h in R/J is f. We would like to find $h_0 \in R^G$ such that $(h)^t - h_0$ for some positive integer t is in R^G. Look at the finite-dimensional, G-invariant subsapce M generated by h^g. [Unfortunately, there is potential for confusion between h^g and $(h)^t$. The first denotes the g-translate of h, the second denotes the t-th power of h. To distinguish between these two, we will put parentheses around h in the latter case.] Since J is invariant, $h^g - h$ is in J for every g. We conclude that $M \cap J$ has codimension 1 in M. We can write every element in M uniquely as $ah + h'$ where $a \in k$ and $h' \in M \cap J$. Sending $ah + h'$ to a defines a G-invariant linear functional l on M.

There is an action of G also on M^*. If we let $h, j_2, \ldots, j_n$ be a basis of M where $j_i \in M \cap J$, we can identify M^* with k^r in terms of the dual basis. The linear functional l corresponds to the vector $(1, 0, \ldots, 0)$. Since G is geometrically reductive, there exists an invariant homogeneous polynomial $F \in k[X_1, \ldots, X_n]$ of degree $t \geq 1$ such that the coefficient of X_1^t does not vanish. Consider the morphism $k[X_1, \ldots, X_n]$ sending X_1 to h and X_i to j_i for $i > 1$.If h_0 is the image of F, $h^t - h_0$ belongs to J. We conclude that $(R/J)^G$ is integral over $R^G/(J \cap R^G)$.

If A is a finitely generated k-algebra which is integral over a subalgebra B, then B is finitely generated. Hence in our case, $R^G/(J \cap R^G)$ is finitely generated. In fact, $(R/J)^G$ is a finite $R^G/(J \cap R^G)$-module.

Choose a non-zero homogeneous element f of R^G of degree at least one. If f is not a zero-divisor, $fR \cap R^G = fR^G$. Since R^G/fR^G is finitely generated, $(R^G/fR^G)_+$ is finitely generated as an ideal. Hence R^G_+ is finitely generated as an ideal in R^G. Hence R^G is a finitely generated k-algebra.

Exercise 6.7. Modify the last paragraph of the proof in case f is a zero-divisor. Hint: Consider the homogeneous ideal I of elements of R that annihilate f. Since $R^G/(fR \cap R^G)$ and $R^G/I \cap R^G$ are both finitely generated, there is a finitely generated subalgebra of R^G that surjects onto both these algebras

In order to handle the non-homogeneous case, we may assume that R^G is a domain. By the homogeneous case S^G is finitely generated. R^G is integral over $S^G/Q \cap S^G$. It suffices to show that the field of fractions of R^G is a finitely generated extension of k. Let T be the set of non-zero divisors of R. Form the ring of fractions of R with respect to T. Let m be the maximal ideal. The field of fractions of R^G may be identified with a subfield of $T^{-1}R/m$. Since $T^{-1}R/m$ is the field of fractions of the finitely generated k-algebra $R/m \cap R$, this follows.

Example 6.8. Everyone's favorite example is the action of $GL(n)$ on the space of $n \times n$ matrices M_n by conjugation. The space of matrices is isomorphic to affine space $\mathbb{A}^{n^2}$. Hence, the coordinate ring is $k[a_{i,j}]$, $1 \leq i, j \leq n$. Any conjugacy class has a representative in Jordan canonical form which is unique upto a permutation of the Jordan blocks. Since the set of eigenvalues of a matrix is invariant under conjugation, we see that the elementary symmetric polynomials of the eigenvalues, i.e. the coefficients of the characteristic polynomial, are invariant under the action. Conversely, suppose that a polynomial is invariant under conjugation. If the eigenvalues are distinct, we can diagonalize the matrix by connjugation. Hence the polynomial must be a symmetric function of the eigenvalues. If the eigenvalues are repeated, the diagonal matrix is in the closure of the orbits with non-trivial Jordan blocks. We conclude that any invariant polynomial is a symmetric polynomial of the eigenvalues. Since the elementary symmetric polynomials generate the ring of symmetric polynomials, we conclude that the ring of invariant functions is generated by the coefficients of the characteristic polynomial.

Now we would like to extend the discussion from actions of reductive groups on affine varieties to actions on projective varieties. Suppose we have a group acting on a projective variety $X \subset \mathbb{P}^n$. A linearization of the action of G is a linear action on k^{n+1} which induces the given action on X. More generally, let X be a variety, G a group acting on it and L a line bundle on X. A linearization of the action of G with respect to L is a linear action on L that induces the action of G on X.

Definition 6.9. A point $x \in X$ is called semi-stable if there exists an invariant homogeneous polynomial that does not vanish on x. A point $x \in X$ is called stable if there exists an invariant polynomial f that does not vanish on x, the action of G on X_f is closed and the dimension of the orbit of x is equal to the dimension of G. These depend not only on the action, but the chosen linearization. Denote the locus of semi-stable points by X^{ss} and the locus of stable points by X^s.

Remark 6.10. Note that the semi-stable points are precisely those that do not contain 0 in the closure of their orbits. Both X^{ss} and X^s are clearly open (possibly empty) in X.

The main theorem of G.I.T. is the existence of a good quotient of the semi-stable locus whose restriction to the stable locus is a geometric quotient. We will call a quotient a good quotient if it satisfies the conditions of Theorem 6.2. We will call a good quotient that is also an orbit space a geometric quotient.

Theorem 6.11. *Let X be a projective variety in $\mathbb{P}^n$. Then for every linear action of a reductive group G on X*

1. *There exists a good quotient (Y, ϕ) of X^{ss} by G and Y is projective.*
2. *There exists an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s$ and (Y^s, ϕ) is a geometric quotient of X^s.*

In view of this theorem it is important to determine the stable and semi-stable loci for reductive group actions on projective varieties. Unfortunately, this in general is a very difficult problem. There is one instance where stability and semi-stability is easy to determine.

Definition 6.12. A one-parameter subgroup is a homomorphism $\lambda : \mathbb{G}_m \to G$.

Any action of k^* on k^{n+1} can be diagonalized. Hence, there exists a basis $e_0, \ldots, e_n$ such that the action of the one-parameter subgroup λ is given by $\lambda(t)e_i =$

$t^{r_i}e_i$ for some integers r_i. If $\hat{x} = \sum x_i e_i$, then

$$\lambda(t)\hat{x} = \sum_i t^{r_i} x_i e_i.$$

Define

$$\mu(x, \lambda) = \max\{-r_i \mid x_i \neq 0\ \}.$$

Theorem 6.13 (The Hilbert-Mumford criterion of stability)**.** *Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Then:*

(1) *x is semi-stable if and only if for every one-parameter subgroup λ of G $\mu(x, \lambda) \geq 0$.*
(2) *x is stable if and only if for every one-parameter subgroup λ of G $\mu(x, \lambda) > 0$.*

Proof. The challenging part of the theorem is to produce a one-parameter subgroup that has the wrong μ invariant if x is not semi-stable. We will sketch Hilbert's proof for the case $G = SL(m)$. The general case follows the same general line of argument (see §2.1 [FKM]).

Let K be the field of fractions of $R = k[[T]]$. If x is not stable, then the morphism $G \to k^{n+1}$ given by sending g to $g\hat{x}$ where $\hat{x}$ is a lift of x is not proper. By the valuative criterion of properness, there exists $\bar{g} \in SL(m, K)$ such that $\bar{g}\hat{x} \in R^{n+1}$, but $\bar{g} \notin SL(m, R)$. We can, however, clear denominators so that $T^r\bar{g} \in SL(m, R)$ for some r. The ring R is a P.I.D., hence we can decompose $\bar{g} = \bar{g_1} d \bar{g_2}$ where g_1 and g_2 are in $SL(m, R)$ and d is a diagonal matrix consisting of entries $T^{w_1}, \dots, T^{w_m}$ for some integers w_i whose sum is zero (since the resulting matrix has to be in $SL(m, K)$. This is the point in the proof where we are using that $G = SL(m)$. To prove the theorem for general groups one needs to use a theorem of Iwahori which asserts that the double coset in $G(R)\backslash G(K)/G(R)$ for a reductive group can be represented by a one-parameter subgroup.

Let g_2 be the matrix obtained by setting $T = 0$ in $\bar{g_2}$. The de-stabilizing one-parameter subgroup is defined by

$$\lambda(t) = g_2^{-1}\mathrm{diag}(t^{w_1}, \dots, t^{w_m})g_2.$$

Diagonalize the action of λ on k^{n+1} with respect to a basis $e_0, \dots, e_n$ as above. We would like to show that if $\hat{x}_i \neq 0$, then the weight r_i of the action on e_i is non-negative. We can also consider the basis $e_0, \dots, e_n$ as a basis of K^{n+1}. Then $g_2^{-1} d g_2 e_i = T^{r_i} e_i$. In particular,

$$g_2^{-1}\bar{g_1}^{-1}\bar{g} = g_2^{-1}\bar{g_1}^{-1}\bar{g_1} d \bar{g_2} = (g_2^{-1} d g_2) g_2^{-1}\bar{g_2}.$$

Therefore, the i-th component of $g_2^{-1}\bar{g_1}^{-1}\bar{g}\hat{x}$ is T^{r_i} times the i-th component of $g_2^{-1}\bar{g_2}\hat{x}$. Consequently, the i-th component of $g_2^{-1}\bar{g_2}\hat{x}$ is in $T^{-r_i}R$. Since it is also in R, we conclude that $r_i \geq 0$. $\square$

Exercise 6.14. Modify the previous argument to obtain the theorem for the semi-stable case.

Example 6.15 (Points on $\mathbb{P}^1$)**.** Consider the action of $SL(2)$ on the homogeneous polynomials of degree d in two variables. Let λ be a one-parameter subgroup of $SL(2)$. If we diagonalize the action of λ on k^2 by $diag(t^a, t^{-a})$ in coordinates (x, y), then the monomials $x^i y^{d-i}$ diagonalizes the action of λ on homogeneous polynomials of degree d. The weight of the action on x^iy^{d-i} is $a(2i-d)$. If we want the weight to be negative, then the coefficient of one monomial x^iy^{d-i} with $2i-d<0$ has to be non-zero. This means that a homogeneous polynomial is stable if and only if it does not have any zeros with multiplicity $\geq d/2$. Similarly, a homogeneous polynomial is semi-stable if and only if it does not have any zeros with multiplicity $> d/2$.

Example 6.16 (Cubic plane curves)**.** Consider the action of $SL(3)$ on the homogeneous polynomials of degree 3 in three variables. If we diagonalize the action of a one-parameter subgroup λ in terms of the coordinates x_1, x_2, x_3 such that $\lambda(t)x_i = t^{w_i}x_i$, then the basis given by monomials $x_1^ix_2^jx_3^{3-i-j}$ diagonalizes the action of λ on degree 3 homogeneous polynomials. The weight of the action on $x_1^ix_2^jx_3^{3-i-j}$ is given by $iw_1+jw_2+(3-i-j)w_3$. We can visualize the one parameter subgroup in terms of barycentric coordinates. The one-parameter subgroups correspond in this picture to lines pivoted around the point $(i,j,3-i-j)=(1,1,1)$. If we move the line without crossing any integral points on the triangle, we do not change the conditions for stability. Also the picture is invariant under the symmetries of the triangle. Analyzing the coefficients we see that a cubic is stable if and only if it is smooth. Similarly a cubic is semi-stable if and only if it has ordinary double points. Note that the G.I.T. quotient of the stable locus in this case constructs the j-line.

Exercise 6.17. Try to generalize the previous example to the action of $SL(3)$ on homogeneous polynomials of degree $4,5,6,....$ In particular, describe what kinds of singularities are allowed on stable curves of degree $4,5,6...$

6.2. **The construction of $\overline{M}_g$.** In view of Theorem 6.11 in order to construct $\overline{M}_g$ we need to show that the N-canonically embedded Deligne-Mumford stable curves are stable points for the $SL(n+1)$-action on the Hilbert scheme and that they form a closed subset. The details of this verification are involved. You may find good accounts in [HM] and [Mum3].

We would like to apply the Hilbert-Mumford criterion to the action of $SL(n+1)$ on $Hilb_{P(m)}(\mathbb{P}^n)$. Fix a one-parameter subgroup λ of $SL(n+1)$. Suppose in terms of homogeneous coordinates x_i that diagonalize the action, the weights are $w_0,\dots,w_n$. Of course, as usual we have that $\sum_i w_i = 0$. Recall that we exhibited the Hilbert scheme as a subscheme of the Grassmannian $G(P(m), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)))$ for m greater than or equal to the regularity of all the ideal sheaves with Hilbert polynomial P. The Grassmannian has natural Plücker coordinates consisting of $P(m)$-element subsets of monomials in the x_i of degree m. This basis also diagonalizes the action of $SL(n+1)$ on $\bigwedge^{P(m)} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$. The weight on the Plücker coordinate $\{\, Y_{j_1},\dots,Y_{j_{P(m)}} \,\}$ where $Y_{j_i} = \prod_r x_r^{m_{j_i,r}}$ is given by

$$\sum_{i,r} w_r m_{j_i,r}.$$

The Hilbert-Mumford criterion for semi-stability then translates to the condition that for each one parameter subgroup, there should be a non-vanishing Plücker coordinate whose weight is non-positive.

We begin by showing that the m-th Hilbert points of smooth, non-degenerate curves embedded by a complete linear series of degree $d \geq 2g$ are stable for the $SL(n+1)$ action.

Theorem 6.18 (Stability for smooth curves)**.** *Let C be a smooth curve of genus $g \geq 2$ embedded in projective space $\mathbb{P}^{d-g}$ by a complete linear system of degree d at least $2g$. Then C is Hilbert stable. Moreover, there exists M such that for all $m \geq M$, the m-th Hilbert point of non-degenerate, smooth curves of degree d and genus g in $\mathbb{P}^{d-g}$ is stable.*

Sketch. The proof is an application of the Hilbert-Mumford criterion. □

Definition 6.19 (Potential stability)**.** A connected curve C of degree d and genus g in $\mathbb{P}^{d-g+1}$ is called potentially stable if

(1) The embedded curve C is non-degenerate.
(2) The abstract curve C is moduli semi-stable.
(3) The linear series embedding C is complete and non-special (i.e. has $h^1 = 0$).
(4) If C' is a complete subcurve of C of arithmetic genus g' meeting the rest of the curve C in k points, then the following estimate holds

$$\left| \deg_{C'}(\mathcal{O}_C(1)) - \frac{d}{g-1}(g_{C'} - 1 + \frac{k}{2}) \right| \leq \frac{k}{2}.$$

Remark 6.20. Observe that if C' is a smooth rational curve meeting the rest of the curve in exactly two points ($k = 2$), then the term $g_{C'} - 1 + k/2 = 0$, hence the degree of C' has to be 1. In other words, C' is a line. By the same argument, if C' is a nodal tree of smooth rational curves meeting the rest of C in exactly two points, then C' is a smooth rational curve since the degree is at most one. Furthermore, C' cannot meet the rest of the curve in only one point.

Recall that $\omega_{C|C'}$ is the dualizing sheaf $\omega_{C'}$ twisted by the nodes connecting C' to C. Hence, $\deg(\omega_{C|C'}) = 2g_{C'} - 2 + k$. Condition (4) has the following alternative useful expression

$$\left| \deg C' - d\frac{\deg(\omega_{C|C'})}{\deg(\omega_C)} \right| \leq \frac{k}{2}.$$

Theorem 6.21 (Potential stability)**.** *Let $g \geq 2$ and $d > 9(g-1)$. Then there is an integer M depending only on d and g such that if $m \geq M$ and $C \in \mathbb{P}^{d-g}$ is a connected curve with semi-stable m-th Hilbert point, then C is potentially stable.*

The proof of this theorem is quite lengthy eventhough the strategy is straightforward. We suppose C has a geometric property that violates potential stability. Under this assumption we construct a one-parameter subgroup that destabilizes the Hilbert point of C contradicting the assumption that the m-th Hilbert point of C was semi-stable.

We first assume Theorem 6.21 and deduce from it the existence of the coarse moduli space $\overline{M}_g$. Fix an integer $r \geq 5$. Consider r-canonically embedded stable curves. Since $\omega_C^{\otimes r}$ is very ample for $r \geq 3$, every Deligne-Mumford stable curve has a representative in the Hilbert scheme $\hat{H} = Hilb_{r(2g-2)+1-g}(\mathbb{P}^{r(2g-2)-g})$. Now consider the subscheme H of $\hat{H}$ subscheme of the Hilbert scheme parameterizing r-canonically embedded Deligne-Mumford stable curves. Let H^{ss} denote the intersection of H with the semi-stable locus of $\hat{H}$. Since $r \geq 5$, we have that the degree of the curves are at least $10(g-1) > 9(g-1)$. Therefore, the assumption of the Potential Stability Theorem is satisfied. We conclude that every semi-stable point of $\hat{H}$ is potentially stable.

Lemma 6.22. *The locus H^{ss} is closed in semi-stable locus of the Hilbert scheme $\hat{H}^{ss}$.*

Proof. To show that H^{ss} is closed we need to show that the inclusion $H^{ss} \to \hat{H}^{ss}$ is proper. By the valuative criterion of properness it suffices to check that given a map from the spectrum of a DVR to $\hat{H}^{ss}$ whose generic point lies in H^{ss}, the closed point also lies in H^{ss}. Given such a map consider the universal curve C_R over *Spec* (R). There are two line bundles on C_R, the relative dualizing sheaf $\omega_{C_R/R}$ and $\mathcal{O}_{C_R}(1)$. These two are isomorphic except possibly at the central fiber. To conclude the lemma we need to show that they also agree on the central fiber. Hence the two differ by $\mathcal{O}_{C_R}(-\sum_i a_i C_i)$ where $\sum_i a_i C_i$ is a linear combination of the central fiber. We need that $a_i = 0$ for all i. We can assume that $a_i \geq 0$ for all i with at least one $a_i = 0$. Let C_1' be the subcurve of the central fiber D where $a_i > 0$ and C_2' be the subcurve of the central fiber D where $a_i = 0$. We see that all $a_i = 0$ as follows. A local equation of $\mathcal{O}_{C_R}(-\sum_i a_i C_i)$ is identically zero on every component of C_2' and on no component of C_1'. In particular, the local equation vanishes at the k points of intersection between C_1' and C_2'. We then have that

$$k \leq \deg_D(\mathcal{O}_{C_R}(-\sum_i a_i C_i) \leq \deg_D(\mathcal{O}_{C_R}(1)) - \frac{\deg_{C_2'}(\mathcal{O}_{C_R}(1)_{|C_2'})}{\deg_{C_2'}(\omega_{|C_2'})} \deg_D(\omega_{|C_2'}) \leq \frac{k}{2}.$$

□

Lemma 6.23. *Every curve C whose Hilbert point lies in H^{ss} is Deligne-Mumford stable.*

Proof. By the potential stability theorem C is semi-stable. In order to show that it is stable we need to check that there are no rational curves that intersect the rest of the curve in only two points. On a rational curve meeting the rest of C in two points, the degree of the dualizing sheaf of C is zero whereas $\mathcal{O}_C(1)$ is very ample. Since these two coincide for points in H^{ss}, we conclude that C must be Deligne-Mumford stable. □

Lemma 6.24. *Every Deligne-Mumford stable curve of genus g has a model in H^{ss}.*

Proof. Every moduli stable curve C is embedded in $\mathbb{P}^{r(2g-2)-g}$ by its $\omega_C^{\otimes r}$. We need to show that the Hilbert point of C lies in H^{ss}. If C is smooth, we already know this by Theorem 6.18. To deduce it for singular Deligne-Mumford stable curves, we take a one-parameter deformation of C to a smooth curve of genus g over the spectrum of a DVR R. If we embed this curve r-canonically, we get a map from *Spec* R to the Hilbert scheme. The generic point lies in H^{ss}. Since the G.I.T. quotient of the Hilbert scheme $\hat{H}^{ss}$ by the action of the special linear group is projective, after a base change we can extend the map to $\hat{H}^{ss}$. Since H^{ss} is closed, the image of the map lies in H^{ss}. Pulling back the universal curve we obtain a semi-stable reduction of a family of stable curves. By the uniqueness of semi-stable reduction this family has to agree with our original family. Since the curves H^{ss} are actually stable, the central fiber of both families have to be projectively equivalent. The lemma follows. □

Lemma 6.25. *Every curve whose Hilbert point lies in H^{ss} is Hilbert stable.*

Proof. We need to show that every point in H^{ss} has closed orbit and the stabilizer of a point in H^{ss} is finite. Suppose the stabilizer is not finite, then the curve

C would have infinitely many automorphisms contradicting that Deligne-Mumford stable curves have only finitely many automorphisms. If the orbit is not closed, then the closure would contain a semi-stable orbit with positive dimensional stabilizer. Again we would obtain a contradiction. □

Lemma 6.26. *The locus H^{ss} is non-singular.*

Proof. Recall that given a Deligne-Mumford stable curve C, there exists a formal scheme $\tilde{C}$ proper and flat over $Spec\ k[[t_1, \dots, t_r]]$ where $r = \dim \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$ such that the special fiber is isomorphic to C. Moreover, for a stable curve the versal deformation is universal and algebrizable and the generic fiber is smooth.

Let $[C] \in H^{ss}$ be a point. Let $\tilde{C}$ be the universal formal deformation of C over $B = Spec\ k[[t_1, \dots, t_r]]$. Set S be the formal completion of H^{ss} at $[C]$. By the universal property of the Hilbert scheme we get a map $S \to H^{ss}$. By the universal property there exists a unique morphism $f : S \to B$ such that the pull-back of the universal curve is $S \times_B \tilde{C}$. The Lemma follows from the claim that $f : S \to B$ is formally smooth. □

One important aspect of the G.I.T. construction is that the projectivity of $\overline{M}_g$ is immediate. Another important consequence is the irreducibility of the moduli space of curves over an algebraically closed field of any characteristic. Originally Deligne and Mumford developed the theory of Deligne-Mumford stacks to prove the irreducibility in all characteristics and for all genus in [DM].

Theorem 6.27. *The moduli space $\overline{M}_g$ is projective.*

Theorem 6.28. *The moduli space $\overline{M}_g$ is irreducible (and reduced) over any algebraically closed field.*

Proof. Soon we will see that the moduli space of curves in characteristic zero is irreducible. There are many ways of seeing this. We will use Teichmüller theory to construct M_g as the quotient of a bounded, contractible domain in $\mathbb{C}^{3g-3}$. Alternatively, one can exhibit every smooth curves as a branched cover of $\mathbb{P}^1$. When the number of branch points is large relative to the degree of the map, using the combinatorics of the symmetric group one may show that the space of branched covers of $\mathbb{P}^1$ is irreducible. Suppose now that the characteristic of the field k is positive. Let R be a discrete valuation ring whose quotient field has characteristic zero and whose residue field is k. The construction outlined so far works over $Spec\ R$. Since the generic fiber of $H^{ss}_R/\mathbb{P}GL \to Spec\ R$ is connnected, by Zariski's connectedness theorem $H^{ss}_R/\mathbb{P}GL \otimes k$ is connected. Since this is an orbit space H^{ss}_k is connected. Since it is smooth, it is reduced and irreducible. Consequently $\overline{M}_g$ is also irreducible. $\overline{M}_g$ is also reduced because the structure sheaf of the quotient is the sheaf of invariants of the structure sheaf of H^{ss}. □

Finally we enumerate the steps that one carries out in order to prove the Potential Stability Theorem. We assume that a geometric condition violating potential stability occurs on a curve. We then produce a one-parameter subgroup destabilizing that point, hence showing that it is not a Hilbert stable point. Unfortunately the number of cases and calculations needed to give a complete proof is rather large. Since we will not use these techniques later in the course, we will just sketch a few sample cases. A complete proof can be found on pages 35-87 of [G].

Claim 6.29. *The first claim is that if a curve C is Hilbert stable, then C_{red} is not contained in a hyperplane.*

If the curve is degenerate, then the map $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(C_{red}, O_{C_{red}}(1))$ has non-trivial kernel. Use the filtration that assigns weight -1 to sections vanishing on C_{red} and weight $w > 0$ to the others so that the average weight is 0. There exists an integer q such that the q-th power of the ideal sheaf of nilpotents in $\mathcal{O}_C$ is zero. Hence no monomial that contains more than q factors of weight -1 can be zero. Provided we choose m such that $(m-q)w > q$, every element of a monomial basis of $H^0(C, \mathcal{O}_C(m))$ has positive weight. Hence, C is not Hilbert semi-stable. From now on we may assume that the linear span of our curves in $\mathbb{P}^n$. This argument is the blueprint for the other arguments. We will give very few details for the other ones.

Claim 6.30. *The second claim is that every component of C is generically reduced.*

Claim 6.31. *The third claim is that every singularity of C_{red} is a double point.*

If p is a point of multiplicity 3 or more, the two-step filtration assigning weight 0 to the sections vanishing at p and weight one to the others is destabilizing.

Claim 6.32. *Every double point of C_{red} is a node.*

Claim 6.33. $H^1(C_{red}, \mathcal{O}_C(1)) = 0$

Claim 6.34. *C is reduced.*

From these claims it follows that the first three conditions of the definition of potential stability hold. The final step is to show that the estimate in (4) holds. This is done by showing that if not the filtration $F_{C'}$ is destabilizing.

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