CHAPTER II

LIE GROUPS AND LIE ALGEBRAS

A Lie group is, roughly speaking, an analytic manifold with a group structure such that the group operations are analytic. Lie groups arise in a natural way as transformation groups of geometric objects. For example, the group of all affine transformations of a connected manifold with an affine connection and the group of all isometries of a pseudo-Riemannian manifold are known to be Lie groups in the compact open topology. However, the group of all diffeomorphisms of a manifold is too big to form a Lie group in any reasonable topology.

The tangent space $\mathfrak{g}$ at the identity element of a Lie group G has a rule of composition $(X,Y) \rightarrow [X,Y]$ derived from the bracket operation on the left invariant vector fields on G. The vector space $\mathfrak{g}$ with this rule of composition is called the Lie algebra of G. The structures of $\mathfrak{g}$ and G are related by the exponential mapping exp: $\mathfrak{g} \rightarrow$ G which sends straight lines through the origin in $\mathfrak{g}$ onto one-paramater subgroups of G. Several properties of this mapping are developed already in §1 because they can be derived as special cases of properties of the Exponential mapping for a suitable affine connection on G. Although the structure of $\mathfrak{g}$ is determined by an arbitrary neighborhood of the identity element of G, the exponential mapping sets up a far-reaching relationship between $\mathfrak{g}$ and the group G in the large. We shall for example see in Chapter VII that the center of a compact simply connected Lie group G is explicitly determined by the Lie algebra $\mathfrak{g}$. In §2 the correspondence (induced by exp) between subalgebras and subgroups is developed. This correspondence is of basic importance in the theory in spite of its weakness that the subalgebra does not in general decide whether the corresponding subgroup will be closed or not, an important distinction when coset spaces are considered.

In §4 we investigate the relationship between homogeneous spaces and coset spaces. It is shown that if a manifold M has a separable transitive Lie transformation group G acting on it, then M can be identified with a coset space G/H (H closed) and therefore falls inside the realm of Lie group theory. Thus, one can, for example, conclude that if H is compact, then M has a G-invariant Riemannian structure.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. If $\sigma \in G$, the inner automorphism $g \rightarrow \sigma g \sigma^{-1}$ induces an automorphism Ad (σ) of $\mathfrak{g}$ and the mapping $\sigma \rightarrow$ Ad (σ) is an analytic homomorphism of G onto an analytic subgroup Ad (G) of $GL(\mathfrak{g})$, the adjoint group. The group Ad (G) can be defined by $\mathfrak{g}$ alone and since its Lie algebra is isomorphic to $\mathfrak{g}/\mathfrak{z}$ ($\mathfrak{z}$ = center of $\mathfrak{g}$), one can, for example, conclude that a semisimple Lie algebra over $\boldsymbol{R}$ is isomorphic to the Lie algebra of a Lie group. This fact holds for arbitrary Lie algebras over $\boldsymbol{R}$ but will not be needed in this book in that generality.

Section 6 deals with some preliminary results about semisimple Lie groups. The main result is Weyl's theorem stating that the universal covering group of a compact semisimple Lie group is compact. In §7 we discuss invariant forms on G and their determination from the structure of $\mathfrak{g}$.

§ 1. The Exponential Mapping

1. The Lie Algebra of a Lie Group

Definition. A *Lie group* is a group G which is also an analytic manifold such that the mapping $(\sigma, \tau) \to \sigma\tau^{-1}$ of the product manifold $G \times G$ into G is analytic.

Examples. 1. Let G be the group of all isometries of the Euclidean plane $\boldsymbol{R}^2$ which preserve the orientation. If $\sigma \in G$, let $(x(\sigma), y(\sigma))$ denote the coordinates of the point $\sigma \cdot 0$ ($0 =$ origin of $\boldsymbol{R}^2$) and let $\theta(\sigma)$ denote the angle between the x-axis l and the image of l under σ. Then the mapping $\varphi : \sigma \to (x(\sigma), y(\sigma), \theta(\sigma))$ maps G in a one-to-one fashion onto the product manifold $\boldsymbol{R}^2 \times S^1$ ($S^1 = \boldsymbol{R} \bmod 2\pi$). We can turn G into an analytic manifold by requiring φ to be an analytic diffeomorphism. An elementary computation shows that for $\sigma, \tau \in G$

$$\begin{aligned}
x(\sigma\tau^{-1}) &= x(\sigma) - x(\tau)\cos(\theta(\sigma) - \theta(\tau)) + y(\tau)\sin(\theta(\sigma) - \theta(\tau));\\
y(\sigma\tau^{-1}) &= y(\sigma) - x(\tau)\sin(\theta(\sigma) - \theta(\tau)) - y(\tau)\cos(\theta(\sigma) - \theta(\tau));\\
\theta(\sigma\tau^{-1}) &= \theta(\sigma) - \theta(\tau) \pmod{2\pi}.
\end{aligned}$$

Since the functions sin and cos are analytic, it follows that G is a Lie group.

2. Let $\tilde{G}$ be the group of all isometries of $\boldsymbol{R}^2$. If s is the symmetry of $\boldsymbol{R}^2$ with respect to a line, then $\tilde{G} = G \cup sG$ (disjoint union). We can turn sG into an analytic manifold by requiring the mapping $\sigma \to s\sigma$ ($\sigma \in G$) to be an analytic diffeomorphism of G onto sG. This makes $\tilde{G}$ a Lie group.

On the other hand, if G_1 and G_2 are two components of a Lie group G and $x_1 \in G_1$, $x_2 \in G_2$, then the mapping $g \to x_2x_1^{-1}g$ is an analytic diffeomorphism of G_1 onto G_2.

Let G be a connected topological group. A *covering group* of G is a pair $(\tilde{G}, \pi)$ where $\tilde{G}$ is a topological group and π is a homomorphism of $\tilde{G}$ into G such that $(\tilde{G}, \pi)$ is a covering space of G. In the case when G is a Lie group, then $\tilde{G}$ has clearly an analytic structure such that $\tilde{G}$ is a Lie group, π analytic and $(\tilde{G}, \pi)$ a covering manifold of G.

Definition. A homomorphism of a Lie group into another which is also an analytic mapping is called an *analytic homomorphism*. An isomorphism of one Lie group onto another which is also an analytic diffeomorphism is called an *analytic isomorphism*.

Let G be a Lie group. If $\rho \in G$, the left translation $L_\rho : g \to \rho g$ of G onto itself is an analytic diffeomorphism. A vector field Z on G is called left invariant if $dL_\rho Z = Z$ for all $\rho \in G$. Given a tangent vector $X \in G_e$ there exists exactly one left invariant vector field $\tilde{X}$ on G such that $\tilde{X}_e = X$ and this $\tilde{X}$ is analytic. In fact, $\tilde{X}$ can be defined by

$$[\tilde{X}f](\rho) = Xf^{L_{\rho^{-1}}} = \left\{\frac{d}{dt}f(\rho\gamma(t))\right\}_{t=0}$$

if $f \in C^\infty(G)$, $\rho \in G$, and $\gamma(t)$ is any curve in G with tangent vector X for $t = 0$. If $X, Y \in G_e$, then the vector field $[\tilde{X}, \tilde{Y}]$ is left invariant due to Prop. 3.3, Chapter I. The tangent vector $[\tilde{X}, \tilde{Y}]_e$ is denoted by $[X, Y]$. The vector space G_e, with the rule of composition $(X, Y) \to [X, Y]$ we denote by $\mathfrak{g}$ (or $\mathfrak{L}(G)$) and call the *Lie algebra of* G.

More generally, let $\mathfrak{a}$ be a vector space over a field K (of characteristic 0). The set $\mathfrak{a}$ is called a *Lie algebra over* K if there is given a rule of composition $(X, Y) \to [X, Y]$ in $\mathfrak{a}$ which is bilinear and satisfies (a) $[X, X] = 0$ for all $X \in \mathfrak{a}$; (b) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{a}$. The identity (b) is called the *Jacobi identity*.

The Lie algebra of G above is clearly a Lie algebra over $\boldsymbol{R}$.

If $\mathfrak{a}$ is a Lie algebra over K and $X \in \mathfrak{a}$, the linear transformation $Y \to [X, Y]$ of $\mathfrak{a}$ is denoted by $\operatorname{ad}X$ (or $\operatorname{ad}_{\mathfrak{a}}X$ when a confusion would otherwise be possible). Let $\mathfrak{b}$ and $\mathfrak{c}$ be two vector subspaces of $\mathfrak{a}$. Then $[\mathfrak{b}, \mathfrak{c}]$ denotes the vector subspace of $\mathfrak{a}$ generated by the set of elements $[X, Y]$ where $X \in \mathfrak{b}$, $Y \in \mathfrak{c}$. A vector subspace $\mathfrak{b}$ of $\mathfrak{a}$ is called a *subalgebra* of $\mathfrak{a}$ if $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}$ and an *ideal* in $\mathfrak{a}$ if $[\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{b}$. If $\mathfrak{b}$ is an ideal in $\mathfrak{a}$ then the factor space $\mathfrak{a}/\mathfrak{b}$ is a Lie algebra with the bracket operation inherited from $\mathfrak{a}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras over the same field K and σ a linear mapping of $\mathfrak{a}$ into $\mathfrak{b}$. The mapping σ is called a *homomorphism* if $\sigma([X, Y]) = [\sigma X, \sigma Y]$ for all $X, Y \in \mathfrak{a}$. If σ is a homomorphism then $\sigma(\mathfrak{a})$ is a subalgebra of $\mathfrak{b}$ and the kernel $\sigma^{-1}\{0\}$ is an ideal in $\mathfrak{a}$. If $\sigma^{-1}\{0\} = \{0\}$, then σ is called an *isomorphism* of $\mathfrak{a}$ into $\mathfrak{b}$. An isomorphism of a Lie algebra onto itself is called an *automorphism*. If $\mathfrak{a}$ is a Lie algebra and $\mathfrak{b}$, $\mathfrak{c}$ subsets of $\mathfrak{a}$, the *centralizer* of $\mathfrak{b}$ in $\mathfrak{c}$ is $\{X \in \mathfrak{c} : [X, \mathfrak{b}] = 0\}$. If $\mathfrak{b} \subset \mathfrak{a}$ is a subalgebra, its *normalizer* in $\mathfrak{a}$ is $\mathfrak{n} = \{X \in \mathfrak{a} : [X, \mathfrak{b}] \subset \mathfrak{b}\}$; $\mathfrak{b}$ is an ideal in $\mathfrak{n}$.

Let V be a vector space over a field K and let $\mathfrak{gl}(V)$ denote the vector space of all endomorphisms of V with the bracket operation $[A, B] = AB - BA$. Then $\mathfrak{gl}(V)$ is a Lie algebra over K. Let $\mathfrak{a}$ be a Lie algebra over K. A homomorphism of $\mathfrak{a}$ into $\mathfrak{gl}(V)$ is called a *representation* of $\mathfrak{a}$ on V. In particular, since $\operatorname{ad}([X, Y]) = \operatorname{ad} X \operatorname{ad} Y - \operatorname{ad} Y \operatorname{ad} X$, the linear mapping $X \to \operatorname{ad} X$ $(X \in \mathfrak{a})$ is a representation of $\mathfrak{a}$ on $\mathfrak{a}$. It is called the *adjoint representation* of $\mathfrak{a}$ and is denoted ad (or $\operatorname{ad}_{\mathfrak{a}}$ when a confusion would otherwise be possible). The kernel of $\operatorname{ad}_{\mathfrak{a}}$ is called the *center* of $\mathfrak{a}$. If the center of $\mathfrak{a}$ equals $\mathfrak{a}$, $\mathfrak{a}$ is said to be *abelian*. Thus $\mathfrak{a}$ is abelian if and only if $[\mathfrak{a}, \mathfrak{a}] = \{0\}$.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras over the same field K. The vector space $\mathfrak{a} \times \mathfrak{b}$ becomes a Lie algebra over K if we define

$$[(X, Y), (X', Y')] = ([X, X'], [Y, Y']).$$

This Lie algebra is called the *Lie algebra product* of $\mathfrak{a}$ and $\mathfrak{b}$. The sets $\{(X, 0) : X \in \mathfrak{a}\}$, $\{(0, Y) : Y \in \mathfrak{b}\}$ are ideals in $\mathfrak{a} \times \mathfrak{b}$ and $\mathfrak{a} \times \mathfrak{b}$ is the direct sum of these ideals.

In the following a Lie algebra shall always mean a finite-dimensional Lie algebra unless the contrary is stated.

2. The Universal Enveloping Algebra

Let $\mathfrak{a}$ be a Lie algebra over a field K. The rule of composition $(X, Y) \to [X, Y]$ is rarely associative; we shall now assign to $\mathfrak{a}$ an associative algebra with unit, the *universal enveloping algebra* of $\mathfrak{a}$, denoted $U(\mathfrak{a})$. This algebra is defined as the factor algebra $T(\mathfrak{a})/J$ where $T(\mathfrak{a})$ is the tensor algebra over $\mathfrak{a}$ (considered as a vector space) and J is the two-sided ideal in $T(\mathfrak{a})$ generated by the set of all elements of the form $X \otimes Y - Y \otimes X - [X, Y]$ where $X, Y \in \mathfrak{a}$. If $X \in \mathfrak{a}$, let X^* denote the image of X under the canonical mapping π of $T(\mathfrak{a})$ onto $U(\mathfrak{a})$. The identity element in $U(\mathfrak{a})$ will be denoted by 1. Then $1 \neq 0$ if $\mathfrak{a} \neq \{0\}$. Proposition 1.9 (b) motivates the consideration of $U(\mathfrak{a})$.

Proposition 1.1. *Let V be a vector space over K. There is a natural one-to-one correspondence between the set of all representations of $\mathfrak{a}$ on V and the set of all representations of $U(\mathfrak{a})$ on V. If ρ is a representation of $\mathfrak{a}$ on V and ρ^* is the corresponding representation of $U(\mathfrak{a})$ on V, then*

$$\rho(X) = \rho^*(X^*) \qquad \text{for } X \in \mathfrak{a}. \tag{1}$$

Proof. Let ρ be a representation of $\mathfrak{a}$ on V. Then there exists a unique representation $\tilde{\rho}$ of $T(\mathfrak{a})$ on V satisfying $\tilde{\rho}(X) = \rho(X)$ for all $X \in \mathfrak{a}$. The mapping $\tilde{\rho}$ vanishes on the ideal J because

$$\tilde{\rho}(X \otimes Y - Y \otimes X - [X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X) - \rho([X, Y]) = 0.$$

Thus we can define a representation ρ^* of $U(\mathfrak{a})$ on V by the condition $\rho^* \circ \pi = \tilde{\rho}$. Then (1) is satisfied and determines ρ^* uniquely. On the other hand, suppose σ is a representation of $U(\mathfrak{a})$ on V. If $X \in \mathfrak{a}$ we put $\rho(X) = \sigma(X^*)$. Then the mapping $X \to \rho(X)$ is linear and in fact a representation of $\mathfrak{a}$ on V, because

$$\begin{aligned}\rho([X, Y]) &= \sigma([X, Y]^*) = \sigma(\pi(X \otimes Y - Y \otimes X)) \\ &= \sigma(X^*Y^* - Y^*X^*) = \rho(X)\rho(Y) - \rho(Y)\rho(X)\end{aligned}$$

for $X, Y \in \mathfrak{a}$. This proves the proposition.

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{a}$ and put $X^*(t) = \sum_{i=1}^n t_i X_i^*$ $(t_i \in K)$. Let $M = (m_1, \ldots, m_n)$ be an ordered set of integers $m_i \geqslant 0$. We shall call M a *positive integral n-tuple.* We put $|M| = m_1 + \ldots + m_n$, $t^M = t_1^{m_1} \ldots t_n^{m_n}$. Considering $t_1, \ldots, t_n$ as indeterminates the various t^M are linearly independent over K and for $|M| > 0$ we can define $X^*(M) \in U(\mathfrak{a})$ as the coefficient of t^M in the expansion of $(|M|!)^{-1}(X^*(t))^{|M|}$. Put $X^*(M) = 1$ if $|M| = 0$.

Proposition 1.2. *The smallest vector subspace of $U(\mathfrak{a})$ containing all the elements $X^*(M)$ (where M is a positive integral n-tuple) is $U(\mathfrak{a})$ itself.*

Proof. It suffices to prove that each element $X_{i_1}^* X_{i_2}^* \ldots X_{i_p}^*$ $(1 \leqslant i_1, \ldots, i_p \leqslant n)$ can be expressed as a finite sum $\sum_{|M| \leqslant p} a_M X^*(M)$ where $a_M \in K$. Consider the element

$$u_p = \frac{1}{p!} \sum_\sigma X_{i_{\sigma(1)}}^* \ldots X_{i_{\sigma(p)}}^*,$$

where σ runs over the permutations of $\{1, \ldots, p\}$. We claim that

$$u_p = m_1! \cdots m_n! \;X^*(M),$$

where m_k is the number of entries in the sequence $(i_1, \ldots, i_p)$ which equal k and $M = (m_1, \ldots, m_n)$. To see this write X_j for X_j^* and note that

$$(t_{i_1} X_{i_1} + \cdots + t_{i_p} X_{i_p})^p = \sum_{|M|=p} t^M S_M\,,$$

where

$$S_M = \sum_\sigma X_{i_{\sigma(1)}} \ldots X_{i_{\sigma(p)}} = p! u_p\,.$$

In each term m_k factors equal X_k. One term is $X_1^{m_1} \cdots X_n^{m_n}$ and the others are obtained by shuffling. In the sum $\sum_\sigma$ each term will appear $m_1! \ldots m_n!$ times. Now $(t_1 X_1 + \cdots t_n X_n)^p = p! \sum_{|M|=p} t^M X(M)$.

Here $p! X(M)$ is the sum of the terms obtained from $X_1^{m_1} \cdots X_n^{m_n}$ by shuffling, each term appearing *exactly once*. Hence

$$p! u_p = m_1! \ldots m_n! \; p! \; X(M)\,.$$

Using the relation $X_j^* X_k^* - X_k^* X_j^* = [X_j, X_k]^*$ we see that

$$X^*_{i_1} \dots X^*_{i_p} - X^*_{i_{\sigma(1)}} \dots X^*_{i_{\sigma(p)}}$$

is a linear combination (with coefficients in K) of elements of the form $X^*_{j_1} \dots X^*_{j_{p-1}}$ $(1 \leqslant j_1 \dots j_{p-1} \leqslant n)$ where each X_{j_q} $(1 \leqslant q \leqslant p-1)$ belongs to the subalgebra of $\mathfrak{a}$ generated by $X_{i_1}, \dots, X_{i_p}$. The formula

$$X^*_{i_1} X^*_{i_2} \dots X^*_{i_p} = \sum_{|M| \leqslant p} a_M X^*(M)$$

now follows by induction on p.

Corollary 1.3. *Let $\mathfrak{b}$ be a subalgebra of $\mathfrak{a}$. Suppose $\mathfrak{b}$ has dimension $n-r$ and let the basis $X_1, \dots, X_n$ of $\mathfrak{a}$ be chosen in such a way that the $n-r$ last elements lie in $\mathfrak{b}$. Let $\mathfrak{B}$ denote the vector subspace of $U(\mathfrak{a})$ spanned by all elements $X^*(M)$ where M varies over all positive integral n-tuples of the form $(0, \dots, 0, m_{r+1}, \dots, m_n)$. Then $\mathfrak{B}$ is a subalgebra of $U(\mathfrak{a})$.*

In fact, the proof above shows that the product $X^*_{i_1} \dots X^*_{i_p}$ $(r < i_1, \dots, i_p \leqslant n)$ can be written as a linear combination of elements $X^*(M)$ for which $m_1 = \dots = m_r = 0$.

3. Left Invariant Affine Connections

Let G be a Lie group, and ∇ an affine connection on G; ∇ is said to be *left invariant* if each L_σ $(\sigma \in G)$ is an affine transformation of G. Let $X_1, \dots, X_n$ be a basis of the Lie algebra $\mathfrak{g}$ of G and let $\tilde{X}_1, \dots, \tilde{X}_n$ denote the corresponding left invariant vector fields on G. Then if ∇ is left invariant, the vector fields $\nabla_{\tilde{X}_i}(\tilde{X}_j)$ $(1 \leqslant i, j \leqslant n)$ are obviously left invariant. On the other hand we can define an affine connection ∇ on G by requiring the $\nabla_{\tilde{X}_i}(\tilde{X}_j)$ to be any left invariant vector fields. Let Z, Z' be arbitrary vector fields in $\mathfrak{D}^1$. Then $Z = \sum_i f_i \tilde{X}_i$, $Z' = \sum_j g_j \tilde{X}_j$ where $f_i, g_j \in C^\infty(G)$. Using the axioms ∇_1 and ∇_2 and Prop. 3.3 in Chapter I we find easily that $\nabla_{dL_\sigma Z}(dL_\sigma Z') = dL_\sigma \nabla_Z(Z')$ for each $\sigma \in G$ so ∇ is left invariant.

Proposition 1.4. *There is a one-to-one correspondence between the set of left invariant affine connections ∇ on G and the set of bilinear functions α on $\mathfrak{g} \times \mathfrak{g}$ with values in $\mathfrak{g}$ given by*

$$\alpha(X, Y) = (\nabla_{\tilde{X}}(\tilde{Y}))_e.$$

Let $X \in \mathfrak{g}$. The following statements are then equivalent:

(i) $\alpha(X, X) = 0$;

(ii) *The geodesic $t \to \gamma_X(t)$ is an analytic homomorphism of $\boldsymbol{R}$ into G.*

Proof. Given a bilinear mapping $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, we define the affine connection ∇ by the requirement

$$\nabla_{\tilde{X}_i}(\tilde{X}_j) = \alpha(X_i, X_j)^\sim \qquad (1 \leqslant i, j \leqslant n).$$

By the remark above, ∇ is left invariant, and the correspondence follows. Also, ∇ is analytic.

Next let $X \in \mathfrak{g}$ and let $\tilde{X}$ be the corresponding left invariant vector field on G. Locally there exist integral curves to the vector field $\tilde{X}$ (Chapter I, §7). In other words, there exists a number $\epsilon > 0$ and a curve segment $\Gamma : t \to \Gamma(t)$ $(0 \leqslant t \leqslant \epsilon)$ in G such that

$$\Gamma(0) = e, \qquad \dot{\Gamma}(s) = \tilde{X}_{\Gamma(s)} \tag{2}$$

for $0 \leqslant s \leqslant \epsilon$. Using induction we define $\Gamma(t)$ for all $t \geqslant 0$ by the requirement

$$\Gamma(t) = \Gamma(n\epsilon)\,\Gamma(t - n\epsilon), \qquad \text{if} \qquad n\epsilon \leqslant t \leqslant (n+1)\,\epsilon,$$

n being a nonnegative integer. On the interval $n\epsilon \leqslant t \leqslant (n+1)\,\epsilon$ we have $\Gamma \circ L_{-n\epsilon} = L_{\Gamma(n\epsilon)^{-1}} \circ \Gamma$. We use both sides of this equation on the tangent vector $(d/dt)_t$ $(n\epsilon \leqslant t \leqslant (n+1)\,\epsilon)$. From (2) we obtain

$$\begin{aligned}\dot{\Gamma}(t) = d\Gamma\left(\frac{d}{dt}\right)_t &= dL_{\Gamma(n\epsilon)} \circ d\Gamma \circ dL_{-n\epsilon}\left(\frac{d}{dt}\right)_t \\ &= dL_{\Gamma(n\epsilon)} \cdot \tilde{X}_{\Gamma(t-n\epsilon)} \\ &= \tilde{X}_{\Gamma(t)}.\end{aligned}$$

Thus (2) holds for all $s \geqslant 0$ (including the points $n\epsilon$).

Assume now $\alpha(X, X) = 0$. Then, due to the left invariance of the corresponding affine connection ∇, we have $\nabla_{\tilde{X}}(\tilde{X}) = 0$. Hence the curve segment $\Gamma(t)$ $(t \geqslant 0)$ is a geodesic segment, and by the uniqueness of such, we have $\Gamma(t) = \gamma_X(t)$ for all $t \geqslant 0$. For any affine connection, $\gamma_{-X}(t) = \gamma_X(-t)$. Since $\alpha(-X, -X) = 0$, it follows that $\gamma_X(t)$ is defined for all $t \in \boldsymbol{R}$. Now let $s \geqslant 0$. Then the curves $t \rightarrow \gamma_X(s+t)$ and $t \rightarrow \gamma_X(s)\,\gamma_X(t)$ are both geodesics in G (since ∇ is left invariant) passing through $\gamma_X(s)$. These geodesics have tangent vectors $\dot{\gamma}_X(s)$ and $dL_{\gamma_X(s)} \cdot X$, respectively, at the point $\gamma_X(s)$. These are equal since (2) holds for all $s \geqslant 0$. We conclude that

$$\gamma_X(s+t) = \gamma_X(s)\,\gamma_X(t) \tag{3}$$

for $s \geqslant 0$ and all t. Using again $\gamma_{-X}(t) = \gamma_X(-t)$, we see that (3) holds for all s and t. This proves that (i) $\Rightarrow$ (ii).

Suppose now θ is any analytic homomorphism of $\boldsymbol{R}$ into G such that $\dot{\theta}(0) = X$. Then from $\theta(s+t) = \theta(s)\,\theta(t)$, $(t, s \in \boldsymbol{R})$, follows that

$$\theta(0) = e, \qquad \dot{\theta}(s) = \tilde{X}_{\theta(s)} \qquad \text{for all } s \in \boldsymbol{R}. \tag{4}$$

In particular, if γ_X is an analytic homomorphism, we have $\nabla_{\tilde{X}}(\tilde{X}) = 0$ on the curve γ_X; hence $\alpha(X, X) = (\nabla_{\tilde{X}}(\tilde{X}))_e = 0$.

Corollary 1.5. *Let $X \in \mathfrak{g}$. There exists a unique analytic homomorphism θ of $\boldsymbol{R}$ into G such that $\dot{\theta}(0) = X$.*

Proof. Let ∇ be any affine connection on G for which $\alpha(X, X) = 0$. Then $\theta = \gamma_X$ is a homomorphism with the required properties. For the uniqueness we observe that (4), in connection with $\alpha(X, X) = 0$, shows, that any homomorphism θ with the required properties must be a geodesic; by the uniqueness of geodesics (Prop. 5.3, Chapter I), $\theta = \gamma_X$.

Definition. For each $X \in \mathfrak{g}$, we put $\exp X = \theta(1)$ if θ is the homomorphism of Cor. 1.5. The mapping $X \to \exp X$ of $\mathfrak{g}$ into G is called the *exponential mapping*.

We have the formula

$$\exp (t + s) X = \exp tX \exp sX$$

for all $s, t \in R$ and all $X \in \mathfrak{g}$. This follows immediately from the fact that if $\alpha(X, X) = 0$, then $\theta(t) = \gamma_X(t) = \gamma_{tX}(1) = \exp tX$.

Definition. A *one-parameter subgroup* of a Lie group G is an analytic homomorphism of R into G.

We have seen above that the one-parameter subgroups are the mappings $t \to \exp tX$ where X is an element of the Lie algebra.

We see from Prop. 1.4 and the corollary that the exponential mapping agrees with the mapping Exp_e (from Chapter I) for all left invariant affine connections on G satisfying $\alpha(X, X) = 0$ for all $X \in \mathfrak{g}$. The classical examples (Cartan and Schouten [1]) are $\alpha \equiv 0$ (the $(-)$-connection), $\alpha(X, Y) = \frac{1}{2}[X, Y]$ (the (0)-connection) and $\alpha(X, Y) = [X, Y]$ (the $(+)$-connection).

From Theorem 6.1, Chapter I, we deduce the following statement.

Proposition 1.6. *There exists an open neighborhood N_0 of 0 in $\mathfrak{g}$ and an open neighborhood N_e of e in G such that exp is an analytic diffeomorphism of N_0 onto N_e.*

Let $X_1, ..., X_n$ be a basis of $\mathfrak{g}$. The mapping

$$\exp (x_1X_1 + ... + x_nX_n) \to (x_1, ..., x_n)$$

of N_e onto N_0 is a coordinate system on N_e, called a system of *canonical coordinates* with respect to $X_1, ..., X_n$. The set N_e is called a *canonical coordinate neighborhood*. Note that N_0 is not required to be star-shaped.

4. Taylor's Formula and the Differential of the Exponential Mapping

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $X \in \mathfrak{g}$, $g \in G$, and $f \in C^\infty(G)$. Since the homomorphism $\theta(t) = \exp tX$ satisfies $\dot\theta(0) = X$ we obtain

$$\tilde{X}_g f = X(f \circ L_g) = \left\{\frac{d}{dt} f(g \exp tX)\right\}_{t=0}, \tag{5}$$

It follows that the value of $\tilde{X}f$ at $g \exp uX$ is

$$[\tilde{X}f](g \exp uX) = \left\{\frac{d}{dt} f(g \exp uX \exp tX)\right\}_{t=0} = \frac{d}{du} f(g \exp uX)$$

and by induction

$$[\tilde{X}^n f](g \exp uX) = \frac{d^n}{du^n} f(g \exp uX).$$

Suppose now that f is analytic at g. Then there exists a star-shaped neighborhood N_0 of 0 in $\mathfrak{g}$ such that

$$f(g \exp X) = P(x_1, \ldots, x_n) \qquad (X \in N_0),$$

where P denotes an absolutely convergent power series and $(x_1, \ldots, x_n)$ are the coordinates of X with respect to a fixed basis of $\mathfrak{g}$. Then we have for a fixed $X \in N_0$

$$f(g \exp tX) = P(tx_1, \ldots, tx_n) = \sum_0^\infty \frac{1}{m!} a_m t^m \qquad (a_m \in R),$$

for $0 \leqslant t \leqslant 1$. It follows that each coefficient a_m equals the mth derivative of $f(g \exp tX)$ for $t = 0$; consequently

$$a_m = [\tilde{X}^m f](g).$$

This proves the "Taylor formula";

$$f(g \exp X) = \sum_0^\infty \frac{1}{n!} [\tilde{X}^n f](g) \tag{6}$$

for $X \in N_0$.

Theorem 1.7. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential mapping of the manifold $\mathfrak{g}$ into G has the differential*

$$d \exp_X = d(L_{\exp X})_e \circ \frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \qquad (X \in \mathfrak{g}).$$

As usual, $\mathfrak{g}$ is here identified with the tangent space $\mathfrak{g}_X$.

Proof:

In the statement, $\frac{1-e^{-A}}{A}$ stands for $\sum_0^\infty \frac{(-A)^n}{(n+1)!}$. For the proof we use the Taylor formula above. We have for f analytic near $\exp X$,

$$(d\exp_X(Y)f)(\exp X) = Y(f \circ \exp)(X)$$

$$= \left\{\frac{d}{dt} f(\exp(X+tY))\right\}_{t=0} = \left\{\frac{d}{dt}\sum_0^\infty \frac{1}{n!}(\widetilde{X}+t\widetilde{Y})^n f(e)\right\}_{t=0}$$

$$= \sum_0^\infty \frac{1}{(n+1)!}(\widetilde{Y}\widetilde{X}^n + \widetilde{X}\widetilde{Y}\widetilde{X}^{n-1} + \cdots + \widetilde{X}^n\widetilde{Y})f(e)\,.$$

Now $(\text{ad } X(Y))^\sim = \widetilde{X}\widetilde{Y} - \widetilde{Y}\widetilde{X}$ so we define

$$\text{ad } X(\widetilde{Y}) = \widetilde{X}\widetilde{Y} - Y\widetilde{X} = L_{\widetilde{X}}\widetilde{Y} - R_{\widetilde{X}}\widetilde{Y}\,.$$

Thus

$$R_{\widetilde{X}} = L_{\widetilde{X}} - \text{ad } X$$

and they commute. Thus

$$\begin{aligned}\widetilde{Y}\widetilde{X}^m &= (R_{\widetilde{X}})^m(\widetilde{Y}) = (L_{\widetilde{X}} - \text{ad } X)^m(\widetilde{Y}) \\ &= \sum_{p=0}^m (-1)^p \binom{m}{p} \widetilde{X}^{m-p}(\text{ad } X)^p(\widetilde{Y})\end{aligned}$$

so

$$\widetilde{Y}\widetilde{X}^n + \cdots + \widetilde{X}^n\widetilde{Y} = \sum_{p=0}^n \widetilde{X}^p\widetilde{Y}\widetilde{X}^{n-p}$$

$$= \sum_{p=0}^n \widetilde{X}^p \sum_{k=0}^{n-p}(-1)^k\binom{n-p}{k}\widetilde{X}^{n-p-k}(\text{ad } X)^k(\widetilde{Y})$$

$$= \sum_{p=0}^n\sum_{k=0}^{n-p}(-1)^k\binom{n-p}{k}\widetilde{X}^{n-k}(\text{ad } X^k)\widetilde{Y})$$

$$= \sum_{k=o}^n\sum_{p=0}^{n-k}(-1)^k\binom{n-p}{k}\widetilde{X}^{n-k}(\text{ad } X)^k(\widetilde{Y})\,.$$

Now what is $\sum_{p=0}^{n-k}\binom{n-p}{k}$? Look at $\sum_{p=0}^{n-k}(1+t)^{n-p}$ and collect coefficient to t^k.

$$\begin{aligned}(1+t)^n+\cdots+(1+t)^k &= (1+t)^k(1+\cdots+(1+t)^{n-k})\\ &= (1+t)^k\frac{(1+t)^{n-k+1}-1}{(1+t)-1}=\frac{(1+t)^{n+1}-(1+t)^k}{t}\end{aligned}$$

and the coefficient to t^k equals $\binom{n+1}{k+1}$. Thus

$$\sum_{p=0}^{n-k}\binom{n-p}{k}=\binom{n+1}{k+1}.$$

so our sum is

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\widetilde{X}^{n-k}(\operatorname{ad}X)^k(\widetilde{Y}).$$

Thus

$$\begin{aligned}(d\exp_X(Y)f)(\exp X) &= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}(\widetilde{X}^{n-k}(\operatorname{ad}X)^k(\widetilde{Y})f)(e)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)!}\frac{1}{(n-k)!}(\widetilde{X}^{n-k}(\operatorname{ad}X)^k(\widetilde{Y})f)(e)\\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^k}{(k+1)!}\frac{1}{(n-k)!}\widetilde{X}^{n-k}(\operatorname{ad}X)^k(\widetilde{Y})f(e)\\ &= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^k}{(k+1)!}\frac{1}{r!}\widetilde{X}^r(\operatorname{ad}X)^k(\widetilde{Y})f(e)\\ &= \sum_{r=0}^{\infty}\frac{\widetilde{X}r}{r!}\sum_{0}^{\infty}\frac{(-1)^k}{(k+1)!}((\operatorname{ad}X)^k(\widetilde{Y})f)(e)\\ &= \left(\frac{1-e^{-\operatorname{ad}X}}{\operatorname{ad}X}(\widetilde{Y})f\right)(\exp X)\\ &= \left(\left(\frac{1-e^{-\operatorname{ad}X}}{\operatorname{ad}X}(Y)\right)^{\sim}f\right)(\exp X)\\ &= dL(\exp X)\circ\frac{1-e^{-\operatorname{ad}X}}{\operatorname{ad}X}(Y)(f)(\exp X).\end{aligned}$$

This proves the formula for f analytic. Thus

$$dL_{\exp(-tX)} \circ d\exp_{tX}(Y) = \frac{1 - e^{-\operatorname{ad} tX}}{\operatorname{ad}(tX)}$$

for all t in some interval $|t| < \delta$. Both sides are analytic functions on $\mathbf{R}$ with values in $\mathfrak{g}$. Since they agree for $|t| < \delta$ they agree on $\mathbf{R}$.

Lemma 1.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let exp be the exponential mapping of $\mathfrak{g}$ into G. Then, if $X, Y \in \mathfrak{g}$,*

(i) $\exp tX \exp tY = \exp\{t(X+Y) + \frac{t^2}{2}[X, Y] + O(t^3)\}$,

(ii) $\exp(-tX)\exp(-tY)\exp tX \exp tY = \exp\{t^2[X, Y] + O(t^3)\}$,

(iii) $\exp tX \exp tY \exp(-tX) = \exp\{tY + t^2[X, Y] + O(t^3)\}$.

In each case $O(t^3)$ denotes a vector in $\mathfrak{g}$ with the property: there exists an $\epsilon > 0$ such that $(1/t^3)\,O(t^3)$ is bounded and analytic for $|t| < \epsilon$.

We first prove (i). Let f be analytic at e. Then using the formula

$$[\tilde{X}^n f](g \exp tX) = \frac{d^n}{dt^n} f(g \exp tX)$$

twice we obtain

$$[\tilde{X}^n \tilde{Y}^m f](e) = \left[\frac{d^n}{dt^n}\frac{d^m}{ds^m} f(\exp tX \exp sY)\right]_{s=0,\,t=0}.$$

Therefore, the Taylor series for $f(\exp tX \exp sY)$ is

$$f(\exp tX \exp sY) = \sum_{m,n \geqslant 0} \frac{t^n}{n!}\frac{s^m}{m!}[\tilde{X}^n \tilde{Y}^m f](e) \tag{7}$$

for sufficiently small t and s. On the other hand,

$$\exp tX \exp tY = \exp Z(t)$$

for sufficiently small t where $Z(t)$ is a function with values in $\mathfrak{g}$, analytic at $t = 0$. We have $Z(t) = tZ_1 + t^2 Z_2 + O(t^3)$ where Z_1 and Z_2 are fixed vectors in $\mathfrak{g}$. Then if f is any of the canonical coordinate functions $\exp(x_1X_1 + \dots + x_nX_n) \to x_i$ we have

$$f(\exp Z(t)) = f(\exp(tZ_1 + t^2Z_2)) + O(t^3)$$

$$= \sum_0^\infty \frac{1}{n!}[(t\tilde{Z}_1 + t^2\tilde{Z}_2)^n f](e) + O(t^3). \tag{8}$$

If we compare (7) for $t = s$ and (8) we find $Z_1 = X + Y$, $\frac{1}{2}\tilde{Z}_1^2 + \tilde{Z}_2 = \frac{1}{2}\tilde{X}^2 + \tilde{X}\tilde{Y} + \frac{1}{2}\tilde{Y}^2$. Consequently

$$Z_1 = X + Y, \qquad Z_2 = \tfrac{1}{2}[X, Y],$$

which proves (i). The relation (ii) is obtained by applying (i) twice. To prove (iii), let again f be analytic at e; then for small t

$$f(\exp tX \exp tY \exp(-tX)) = \sum_{m,n,p \geq 0} \frac{t^m}{m!}\frac{t^n}{n!}\frac{t^p}{p!}[\tilde{X}^m\tilde{Y}^n(-\tilde{X})^p f](e) \tag{9}$$

and

$$\exp tX \exp tY \exp(-tX) = \exp S(t)$$

where $S(t) = tS_1 + t^2S_2 + O(t^3)$ and $S_1, S_2 \in \mathfrak{g}$. If f is any canonical coordinate function, then

$$f(\exp S(t)) = f(\exp(tS_1 + t^2S_2)) + O(t^3)$$

$$= \sum_0^\infty \frac{1}{n!}[(t\tilde{S}_1 + t^2\tilde{S}_2)^n f](e) + O(t^3), \tag{10}$$

and we find by comparing coefficients in (9) and (10), $S_1 = Y$, $S_2 = [X, Y]$, which proves (iii).

Remark. The relation (ii) gives a geometric interpretation of the bracket $[X, Y]$; in fact, it shows that $[X, Y]$ is the tangent vector at e to the C^1 curve segment

$$s \to \exp(-\sqrt{s}X)\exp(-\sqrt{s}Y)\exp\sqrt{s}X\exp\sqrt{s}Y \qquad (s \geqslant 0).$$

Note also that this is a special case of Exercise A.8, Chapter I.

Let $D(G)$ denote the algebra of operators on $C^\infty(G)$ generated by all the left invariant vector fields on G and I (the identity operator on

$C^\infty(G)$). If $X \in \mathfrak{g}$ we shall also denote the corresponding left invariant vector field on G by X. Similarly the operator $\tilde{X}_1 \cdot \tilde{X}_2 \dots \tilde{X}_k$ ($X_i \in \mathfrak{g}$) will be denoted by $X_1 \cdot X_2 \dots X_k$ for simplicity. Let $X_1, \dots, X_n$ be any basis of $\mathfrak{g}$ and put $X(t) = \sum_{i=1}^n t_i X_i$. Let $M = (m_1, \dots, m_n)$ be a positive integral n-tuple, let $t^M = t_1^{m_1} \cdots t_n^{m_n}$ and let $X(M)$ denote the coefficient of t^M in the expansion of $(|M|!)^{-1}(X(t))^{|M|}$. If $|M| = 0$ put $X(M) = I$. It is clear that $X(M) \in \boldsymbol{D}(G)$.

Proposition 1.9.

(a) *As M varies through all positive integral n-tuples the elements $X(M)$ form a basis of $\boldsymbol{D}(G)$ (considered as a vector space over $\boldsymbol{R}$).*

(b) *The universal enveloping algebra $U(\mathfrak{g})$ is isomorphic to $\boldsymbol{D}(G)$.*

Proof. Let f be an analytic function at $g \in G$; we have by (6)

$$f(g \exp X(t)) = \sum_M t^M [X(M)f](g), \tag{11}$$

if the t_i are sufficiently small. If we compare this formula with the ordinary Taylor formula for the function F defined by $F(t_1, \dots, t_n) = f(g \exp X(t))$, we obtain

$$[X(M)f](g) = \frac{1}{m_1! \dots m_n!} \left\{ \frac{\partial^{|M|}}{\partial t_1^{m_1} \dots \partial t_n^{m_n}} f(g \exp X(t)) \right\}_{t_1 = \dots = t_n = 0}. \tag{12}$$

It follows immediately that the various $X(M)$ are linearly independent. The Lie algebra $\mathfrak{g}$ has a representation ρ on $C^\infty(G)$ if we associate to each $X \in \mathfrak{g}$ the corresponding left invariant vector field. The representation ρ^* from Prop. 1.1 gives a homomorphism of $U(\mathfrak{g})$ into $\boldsymbol{D}(G)$ such that $\rho^*(X^*) = \rho(X)$ for $X \in \mathfrak{g}$. The mapping ρ^* sends the element $X_{i_1}^* \dots X_{i_p}^* \in U(\mathfrak{g})$ into $X_{i_1} \dots X_{i_p} \in \boldsymbol{D}(G)$; thus $\rho^*(U(\mathfrak{g})) = \boldsymbol{D}(G)$. Moreover, ρ^* sends the element $X^*(M) \in U(\mathfrak{g})$ into $X(M) \in \boldsymbol{D}(G)$. Since the elements $X(M)$ are linearly independent, the proposition follows from Prop. 1.2.

Corollary 1.10. *With the notation above, the elements $X_1^{e_1} \dots X_n^{e_n}$ ($e_i \geqslant 0$) form a basis of $\boldsymbol{D}(G)$.*

Since $X_i X_j - X_j X_i = [X_i, X_j]$ it is clear that each $X(M)$ can be written as a real linear combination of elements $X_1^{e_1} \dots X_n^{e_n}$ where $e_1 + \dots + e_n \leqslant |M|$. On the other hand, as noted in the proof of Prop. 1.2, each $X_1^{e_1} \dots X_n^{e_n}$ can be written as a real linear combination of elements $X(M)$ for which $|M| \leqslant e_1 + \dots + e_n$. Since the number of elements $X(M)$, $|M| \leqslant e_1 + \dots + e_n$ equals the number of elements

$X_1^{f_1} \dots X_n^{f_n}$ $(f_1 + \dots + f_n \leqslant e_1 + \dots + e_n)$, the corollary follows from Prop. 1.9.

This corollary shows quickly that $D(G)$ has no divisors of 0.

Definition. Let G and G' be two Lie groups with identity elements e and e'. These groups are said to be *isomorphic* if there exists an analytic isomorphism of G onto G'. The groups G and G' are said to be *locally isomorphic* if there exist open neighborhoods U and U' of e and e', respectively, and an analytic diffeomorphism f of U onto U' satisfying:

(a) If $x, y, xy \in U$, then $f(xy) = f(x)f(y)$.

(b) If $x', y', x'y' \in U'$, then $f^{-1}(x'y') = f^{-1}(x')f^{-1}(y')$.

Theorem 1.11. *Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.*

Proof. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$. Owing to Prop. 1.9 we can legitimately write $X(M)$ instead of $X^*(M)$; there exist uniquely determined constants $C^P{}_{MN} \in \boldsymbol{R}$ such that

$$X(M)\,X(N) = \sum_P C^P{}_{MN}\,X(P),$$

M, N, and P denoting positive integral n-tuples. Owing to Prop. 1.9, the constants $C^P{}_{MN}$ depend only on the Lie algebra $\mathfrak{g}$. If N_e is a canonical coordinate neighborhood of $e \in G$ and $g \in N_e$ let $g_1, \dots, g_n$ denote the canonical coordinates of g. Then if $x, y, xy \in N_e$, we have

$$x = \exp(x_1X_1 + \dots + x_nX_n), \qquad y = \exp(y_1X_1 + \dots + y_nX_n),$$
$$xy = \exp((xy)_1X_1 + \dots + (xy)_nX_n).$$

We also put

$$x^M = x_1^{m_1} \dots x_n^{m_n}, \qquad y^M = y_1^{m_1} \dots y_n^{m_n}.$$

Using (7) on the function $f: x \to x_k$ we find for sufficiently small x_i, y_j

$$(xy)_k = \sum_{M,N} x^My^N[X(M)\,X(N)\,x_k]\,(e). \tag{13}$$

From (12) it follows that

$$[X(P)\,x_k]\,(e) = \begin{cases} 1 \text{ if } P = (\delta_{k1}, \delta_{k2}, \dots, \delta_{kn}), \\ 0 \text{ otherwise}. \end{cases}$$

Putting $[k] = (\delta_{1k}, \dots, \delta_{nk})$, relation (13) becomes

$$(xy)_k = \sum_{M,N} C^{[k]}{}_{MN}\,x^My^N, \tag{14}$$

if x_i, y_j $(1 \leqslant i, j \leqslant n)$ are sufficiently small. This last formula shows that the group law is determined in a neighborhood of e by the Lie algebra. In particular, Lie groups with isomorphic Lie algebras are locally isomorphic. Before proving the converse of Theorem 1.11 we prove a general lemma about homomorphisms.

Lemma 1.12. *Let H and K be Lie groups with Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$, respectively. Let φ be an analytic homomorphism of H into K. Then $d\varphi_e$ is a homomorphism of $\mathfrak{h}$ into $\mathfrak{k}$ and*

$$\varphi(\exp X) = \exp d\varphi_e(X) \qquad (X \in \mathfrak{h}). \tag{15}$$

Proof. Let $X \in \mathfrak{h}$. The mapping $t \to \varphi(\exp tX)$ is an analytic homomorphism of $\boldsymbol{R}$ into K. If we put $X' = d\varphi_e(X)$, Cor. 1.5 implies that $\varphi(\exp tX) = \exp tX'$ for all $t \in \boldsymbol{R}$. Since φ is a homomorphism, we have $\varphi \circ L_\sigma = L_{\varphi(\sigma)} \circ \varphi$ for $\sigma \in H$. It follows that

$$(d\varphi_\sigma) \circ dL_\sigma \cdot X = dL_{\varphi(\sigma)} \cdot X'.$$

This means that the left invariant vector fields $\tilde{X}$ and $\tilde{X}'$ are φ-related. Hence, by Prop. 3.3, Chapter I, $d\varphi_e$ is a homomorphism and the lemma is proved.

To finish the proof of Theorem 1.11 we suppose now that the Lie groups G and G' are locally isomorphic. Let $\mathfrak{g}$ and $\mathfrak{g}'$ denote their respective Lie algebras. Suppose φ is a local isomorphism of G into G', and let $X' = d\varphi_e(X)$. If $g \in G$ is sufficiently near e then

$$\varphi \circ L_g = L_{\varphi(g)} \circ \varphi \text{ on a neighborhood of } e.$$

Thus $d\varphi \circ dL_g(X) = dL_{\varphi(g)}(X')$, so in a neighborhood of e , $\widetilde{X}$ and $\widetilde{X}'$ are φ related. If $Y \in \mathfrak{g}$ then $\widetilde{Y}$ and $\widetilde{Y}'$ $(Y' = d\varphi(Y))$ are φ related in a neighborhood of e, and so are $[\widetilde{X}, \widetilde{Y}]$ and $[\widetilde{X}', \widetilde{Y}']$. Thus $d\varphi_e([X,Y]) = [X', Y']$ as claimed.

Example. Let $GL(n, \boldsymbol{R})$ denote the group of all real nonsingular $n \times n$ matrices and let $\mathfrak{gl}(n, \boldsymbol{R})$ denote the Lie algebra of all real $n \times n$ matrices, the bracket being $[A, B] = AB - BA$, $A, B \in \mathfrak{gl}(n, \boldsymbol{R})$. If we consider the matrix $\sigma = (x_{ij}(\sigma)) \in GL(n, \boldsymbol{R})$ as the set of coordinates of a point in $\boldsymbol{R}^{n^2}$ then $GL(n, \boldsymbol{R})$ can be regarded as an open submanifold of $\boldsymbol{R}^{n^2}$. With this analytic structure $GL(n, \boldsymbol{R})$ is a Lie group;

this is obvious by considering the expression of $x_{ij}(\sigma\tau^{-1})$ $(\tau, \sigma \in GL(n, R))$ in terms of $x_{kl}(\sigma)$, $x_{pq}(\tau)$, given by matrix multiplication.

Let X be an element of $\mathfrak{L}(GL(n, R))$ and let $\tilde{X}$ denote the left invariant vector field on $GL(n, R)$ such that $\tilde{X}_e = X$. Let $(a_{ij}(X))$ denote the matrix $(\tilde{X}_e x_{ij})$. We shall prove that the mapping $\varphi : X \to (a_{ij}(X))$ is an isomorphism of $\mathfrak{L}(GL(n, R))$ onto $\mathfrak{gl}(n, R)$. The mapping φ is linear and one-to-one; in fact, the relation $(a_{ij}(X)) = 0$ implies $\tilde{X}_e f = 0$ for all differentiable functions f, hence $\tilde{X} = 0$. Considering the dimensions of the Lie algebras we see that the range of φ is $\mathfrak{gl}(n, R)$. Next we consider $[\tilde{X}x_{ij}](\sigma) = (dL_\sigma X)\, x_{ij} = X(x_{ij} \circ L_\sigma)$. If $\tau \in GL(n, R)$, then

$$(x_{ij} \circ L_\sigma)(\tau) = x_{ij}(\sigma\tau) = \sum_{k=1}^{n} x_{ik}(\sigma)\, x_{kj}(\tau). \tag{16}$$

Hence

$$[\tilde{X}x_{ij}](\sigma) = \sum_{k=1}^{n} x_{ik}(\sigma)\, a_{kj}(X). \tag{17}$$

It follows that

$$[(\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X})\, x_{ij}](e) = \sum_{k=1}^{n} a_{ik}(X)\, a_{kj}(Y) - a_{ik}(Y)\, a_{kj}(X)$$

$$= [\varphi(X), \varphi(Y)]_{ij}.$$

Consequently, the Lie algebra of $GL(n, R)$ can be identified with $\mathfrak{gl}(n, R)$ so we now write X_{ij} instead of $a_{ij}(X)$ above. Using the general formula

$$[\tilde{X}f](\exp tX) = \frac{d}{dt} f(\exp tX)$$

for a differentiable function f, we obtain from (16) and (17)

$$\frac{d}{dt} x_{ij}(\exp tX) = \sum_{k=1}^{n} x_{ik}(\exp tX)\, X_{kj}.$$

Thus the matrix function $Y(t) = \exp tX$ satisfies the differential equation

$$\frac{dY(t)}{dt} = Y(t)\, X, \qquad Y(0) = I.$$

Since this equation is also satisfied by the matrix exponential function

$$Y(t) = e^{tX} = I + tX + \frac{t^2X^2}{2!} + \dots,$$

we conclude that $\exp X = e^X$ for all $X \in \mathfrak{gl}(n, R)$.

Let V be an n-dimensional vector space over $\boldsymbol{R}$. Let $\mathfrak{gl}(V)$ be the Lie algebra of all endomorphisms of V and let $GL(V)$ be the group of invertible endomorphisms of V. Fix a basis $e_1, \ldots, e_n$ of V. To each $\sigma \in \mathfrak{gl}(V)$ we associate the matrix $(x_{ij}(\sigma))$ given by

$$\sigma e_j = \sum_{i=1}^{n} x_{ij}(\sigma)\, e_i.$$

The mapping $J_e : \sigma \to (x_{ij}(\sigma))$ is an isomorphism of $\mathfrak{gl}(V)$ onto $\mathfrak{gl}(n, \boldsymbol{R})$ whose restriction to $GL(V)$ is an isomorphism of $GL(V)$ onto $GL(n, \boldsymbol{R})$. This isomorphism turns $GL(V)$ into a Lie group with Lie algebra isomorphic to $\mathfrak{gl}(V)$. If $f_1, \ldots, f_n$ is another basis of V, we get another isomorphism $J_f : \mathfrak{gl}(V) \to \mathfrak{gl}(n, \boldsymbol{R})$. If $A \in GL(V)$ is determined by $Ae_i = f_i$ $(1 \leqslant i \leqslant n)$, then J_f and J_e are connected by the equation $J_e(\sigma) = J_f(A) J_f(\sigma) J_f(A^{-1})$. Since the mapping $g \to J_f(A) g J_f(A^{-1})$ is an analytic isomorphism of $GL(n,\boldsymbol{R})$ onto itself, we conclude: (1) The analytic structure of $GL(V)$ is independent of the choice of basis. (2) There is an isomorphism of $\mathfrak{L}(GL(V))$ onto $\mathfrak{gl}(V)$ (namely, $J_e^{-1} \circ dJ_e$) which is independent of the choice of basis of V.

§ 2. Lie Subgroups and Subalgebras

Definition. Let G be a Lie group. A submanifold H of G is called a *Lie subgroup* if

(i) H is a subgroup of the (abstract) group G;

(ii) H is a topological group.

A Lie subgroup is itself a Lie group; in order to see this, consider the analytic mapping $\alpha : (x, y) \to xy^{-1}$ of $G \times G$ into G. Let α_H denote the restriction of α to $H \times H$. Then the mapping $\alpha_H : H \times H \to G$ is analytic, and by (ii) the mapping $\alpha_H : H \times H \to H$ is continuous. In view of Lemma 3.4 Chapter I, the mapping α_H is an analytic mapping of $H \times H$ into H so H is a Lie group.

A connected Lie subgroup is often called an *analytic subgroup*.

Theorem 2.1. *Let G be a Lie group. If H is a Lie subgroup of G, then the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$, the Lie algebra of G. Each subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one connected Lie subgroup of G.*

Proof. If I denotes the identity mapping of H into G, then by Lemma 1.12 dI_e is a homomorphism of $\mathfrak{h}$ into $\mathfrak{g}$. Since H is a submanifold of G, dI_e is one-to-one. Thus $\mathfrak{h}$ can be regarded as a subalgebra of $\mathfrak{g}$.

Let $\exp_{\mathfrak{h}}$ and $\exp_{\mathfrak{g}}$, respectively, denote the exponential mappings of $\mathfrak{h}$ into H and of $\mathfrak{g}$ into G. From Cor. 1.5 we get immediately

$$\exp_{\mathfrak{h}}(X) = \exp_{\mathfrak{g}}(X), \qquad X \in \mathfrak{h}. \tag{1}$$

We can therefore drop the subscripts and write exp instead of $\exp_{\mathfrak{h}}$ and $\exp_{\mathfrak{g}}$. If $X \in \mathfrak{h}$, then the mapping $t \to \exp tX$ ($t \in \boldsymbol{R}$) is a curve in H. On the other hand, suppose $X \in \mathfrak{g}$ such that the mapping $t \to \exp tX$ is a path in H, that is, a continuous curve in H. By Lemma 14.1, Chapter I, the mapping $t \to \exp tX$ is an analytic mapping of $\boldsymbol{R}$ into H. Thus $X \in \mathfrak{h}$, so we have

$$\mathfrak{h} = \{X \in \mathfrak{g} : \text{the map } t \to \exp tX \text{ is a path in } H\}. \tag{2}$$

To prove the second statement of Theorem 2.1, suppose $\mathfrak{h}$ is any subalgebra of $\mathfrak{g}$. Let H be the smallest subgroup of G containing $\exp \mathfrak{h}$. Let $(X_1, ..., X_n)$ be a basis of $\mathfrak{g}$ such that (X_i) $(r < i \leqslant n)$ is a basis of $\mathfrak{h}$. Then we know from Cor. 1.3 (and Prop. 1.9) that all real linear combinations of elements $X(M)$, where the n-tuple M has the form $(0, ..., 0, m_{r+1}, ..., m_n)$, actually form a subalgebra of $U(\mathfrak{g})$. Let $|X| = (x_1^2 + ... + x_n^2)^{1/2}$ if $X = x_1X_1 + ... + x_nX_n$ $(x_i \in \boldsymbol{R})$. Choose $\delta > 0$ such that exp is a diffeomorphism of the open ball $B_\delta = \{X : |X| < \delta\}$ onto an open neighborhood N_e of e in G and such that (14), §1, holds for $x, y, xy \in N_e$. Denote the subset $\exp(\mathfrak{h} \cap B_\delta)$ of N_e by V. The mapping

$$\exp(x_{r+1}X_{r+1} + ... + x_nX_n) \to (x_{r+1}, ..., x_n)$$

is a coordinate system on V with respect to which V is a connected manifold. Since $\mathfrak{h} \cap B_\delta$ is a submanifold of B_δ, V is a submanifold of N_e; hence V is a submanifold of G. Now suppose $x, y \in V$, $xy \in N_e$, and consider the canonical coordinates of xy as given by (14), §1. Since $x_k = y_k = 0$, if $1 \leqslant k \leqslant r$, we find (using the remark above about $X(M)$) that $(xy)_k = 0$ if $1 \leqslant k \leqslant r$. Thus we have

$$VV \cap N_e \subset V. \tag{3}$$

Let $\mathscr{V}$ denote the family of subsets of H containing a neighborhood of e in V. Let us verify that $\mathscr{V}$ satisfies the following six axioms for a topological group (Chevalley **Theory of Lie Groups I, Ch. II, § II**)

I. The intersection of any two sets of $\mathscr{V}$ lies in $\mathscr{V}$.

II. The intersection of all sets of $\mathscr{V}$ is $\{e\}$.

III. Any subset of H containing a set in $\mathscr{V}$ lies in $\mathscr{V}$.

IV. If $U \in \mathscr{V}$, there exists a set $U_1 \in \mathscr{V}$ such that $U_1U_1 \subset U$.

V. If $U \in \mathscr{V}$, then $U^{-1} \in \mathscr{V}$.

VI. If $U \in \mathscr{V}$ and $h \in H$, then $hUh^{-1} \in \mathscr{V}$.

Of these axioms I, II, III and V are obvious. For IV let $U \in \mathcal{N}$. Choose $\epsilon < \delta$ such that

$$U \cap \exp(B_\epsilon \cap \mathfrak{h}) = V_\epsilon, \text{ say.}$$

By the proof of (3), $V_\epsilon V_\epsilon \cap \exp B_\epsilon \subset V_\epsilon$. Select $\delta_1 < \epsilon$ such that $\exp B_{\delta_1} \exp B_{\delta_1} \subset \exp B_\epsilon$ and put $U_1 = \exp(\mathfrak{h} \cap B_{\delta_1})$. Then

$$U_1 U_1 \subset V_\epsilon V_\epsilon \cap \exp B_\epsilon \subset V_\epsilon \subset U$$

proving IV.

For VI, let $U \in \mathscr{V}$ and $h \in H$. Let log denote the inverse of the mapping $\exp : B_\delta \to N_e$. Then log maps V onto $\mathfrak{h} \cap B_\delta$. If $X \in \mathfrak{g}$, there exists a unique vector $X' \in \mathfrak{g}$ such that $h \exp tX\, h^{-1} = \exp tX'$ for all $t \in \boldsymbol{R}$. The mapping $X \to X'$ is an automorphism of $\mathfrak{g}$ (Lemma 1.12); it maps $\mathfrak{h}$ into itself as is easily seen from (3) by using a decomposition $h = \exp Z_1 \ldots \exp Z_p$ where each Z_i belongs to $B_\delta \cap \mathfrak{h}$. Consequently, we can select δ_1 $(0 < \delta_1 < \delta)$ such that the open ball B_{δ_1} satisfies

$$h \exp (B_{\delta_1} \cap \mathfrak{h})\, h^{-1} \subset V,$$

$$h (\exp B_{\delta_1})\, h^{-1} \subset N_e.$$

The mapping $X \to \log (h \exp X h^{-1})$ of $B_{\delta_1} \cap \mathfrak{h}$ into $B_\delta \cap \mathfrak{h}$ is regular so the image of $B_{\delta_1} \cap \mathfrak{h}$ is a neighborhood of 0 in $\mathfrak{h}$. Applying the mapping exp we see that $h \exp (B_{\delta_1} \cap \mathfrak{h})\, h^{-1}$ is a neighborhood of e in V. This shows that $hUh^{-1} \in \mathscr{V}$. Axioms I-VI are therefore satisfied. Hence there exists a topology on H such that H is a topological group and such that $\mathscr{V}$ is the family of neighborhoods of e in H. In particular V is a neighborhood of e in H.

For each $z \in G$, consider the mapping

$$\Phi_z : z \exp (x_1 X_1 + \ldots + x_n X_n) \to (x_1, \ldots, x_n),$$

which maps zN_e onto B_δ. Let φ_z denote the restriction of Φ_z to zV. If $z \in H$ then φ_z maps the neighborhood zV (of z in H) onto the open subset $B_\delta \cap \mathfrak{h}$ in Euclidean space $\boldsymbol{R}^{n-r}$. Moreover, if $z_1, z_2 \in H$ the mapping $\varphi_{z_1} \circ \varphi_{z_2}^{-1}$ is the restriction of $\Phi_{z_1} \circ \Phi_{z_2}^{-1}$ to an open subset of $\mathfrak{h}$, hence analytic. The space H with the collection of maps φ_z, $z \in H$, is therefore an analytic manifold.

Now V is a submanifold of G. Since left translations are diffeomorphisms of H it follows that H is a submanifold of G. Hence H is a Lie subgroup of G.

We know that $\dim H = \dim \mathfrak{h}$. For $i > r$ the mapping $t \to \exp tX_i$ is a curve in H. This in view of (2) proves that H has Lie algebra $\mathfrak{h}$. Moreover, H is connected since it is generated by $\exp \mathfrak{h}$ which is a connected neighborhood of e in H.

Finally, in order to prove uniqueness, suppose H_1 is any connected Lie subgroup of G with $(H_1)_e = \mathfrak{h}$. From (1) we see that $H = H_1$ (set theoretically). Since exp is an analytic diffeomorphism of a neighborhood of 0 in $\mathfrak{h}$ onto a neighborhood of e in H and H_1, it is clear that the Lie groups H and H_1 coincide.

Corollary 2.2. *Suppose H_1 and H_2 are two Lie subgroups of a Lie group G such that $H_1 = H_2$ (as topological groups). Then $H_1 = H_2$ (as Lie groups).*

Relation (2) shows, in fact, that H_1 and H_2 have the same Lie algebra. By Theorem 2.1, their identity components coincide as Lie groups. Since left translations on H_1 and H_2 are analytic, it follows at once that the Lie groups H_1 and H_2 coincide.

Theorem 2.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and H an (abstract) subgroup of G. Suppose H is a closed subset of G. Then there exists a unique analytic structure on H such that H is a topological Lie subgroup of G.*

We begin by proving a simple lemma.

Lemma 2.4. *Suppose $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{m} + \mathfrak{n}$ where $\mathfrak{m}$ and $\mathfrak{n}$ are two vector subspaces of $\mathfrak{g}$. Then there exist bounded, open, connected neighborhoods $U_\mathfrak{m}$ and $U_\mathfrak{n}$ of 0 in $\mathfrak{m}$ and $\mathfrak{n}$, respectively, such that the mapping $\Phi : (A, B) \to \exp A \exp B$ is a diffeomorphism of $U_\mathfrak{m} \times U_\mathfrak{n}$ onto an open neighborhood of e in G.*

Proof: The differential $d\Phi_{(0,0)}$ is the identity on $\mathfrak{m}$ and on $\mathfrak{h}$ hence on all of $\mathfrak{g}$. This proves the lemma in view of Prop. 3.1, Chapter I.

Remark. The lemma generalizes immediately to an arbitrary direct decomposition $\mathfrak{g} = \mathfrak{m}_1 + \ldots + \mathfrak{m}_s$ of $\mathfrak{g}$ into subspaces.

Turning now to the proof of Theorem 2.3, let $\mathfrak{h}$ denote the subset of $\mathfrak{g}$ given by

$$\mathfrak{h} = \{X : \exp tX \in H \text{ for all } t \in \boldsymbol{R}\}.$$

We shall prove that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. First we note that $X \in \mathfrak{h}$, $s \in \boldsymbol{R}$ implies $sX \in \mathfrak{h}$. Next, suppose $X, Y \in \mathfrak{h}$. By Lemma 1.8 we have for a given $t \in \boldsymbol{R}$,

$$\left(\exp \frac{t}{n} X \exp \frac{t}{n} Y\right)^n = \exp \left\{t(X + Y) + \frac{t^2}{2n}[X, Y] + O\left(\frac{1}{n^2}\right)\right\},$$

$$\left(\exp\left(-\frac{t}{n} X\right) \exp\left(-\frac{t}{n} Y\right) \exp \frac{t}{n} X \exp \frac{t}{n} Y\right)^{n^2} = \exp \left\{t^2[X, Y] + O\left(\frac{1}{n}\right)\right\}.$$

The left-hand sides of these equations belong to H; since H is closed, the limit as $n \to \infty$ also belongs to H. Thus $t(X + Y) \in \mathfrak{h}$ and $t^2[X, Y] \in \mathfrak{h}$ as desired.

Let H^* denote the connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then $H^* \subset H$. We shall now prove, that if H is given the relative topology of G, and H_0 is the identity component of H, then $H^* = H_0$ (as topological groups). For this we now prove that if N is a neighborhood of e in H^*, then N is a neighborhood of e in H. If N were not a neighborhood of e in H, there would exist a sequence $(c_k) \subset H - N$ such that $c_k \to e$ (in the topology of G). Using Lemma 2.4 for $\mathfrak{h} = \mathfrak{n}$ and $\mathfrak{m}$ any complementary subspace we can assume that $c_k = \exp A_k \exp B_k$ where $A_k \in U_\mathfrak{m}$, $B_k \in U_\mathfrak{n}$, and $\exp B_k \in N$. Then

$$A_k \neq 0, \qquad \lim A_k = 0.$$

Since $A_k \neq 0$, there exists an integer $r_k > 0$ such that

$$r_k A_k \in U_\mathfrak{m}, \qquad (r_k + 1)\, A_k \notin U_\mathfrak{m}.$$

Now, $U_\mathfrak{m}$ is bounded, so we can assume, passing to a subsequence if necessary, that the sequence $(r_k A_k)$ converges to a limit $A \in \mathfrak{m}$. Since $(r_k + 1)\, A_k \notin U_\mathfrak{m}$ and $A_k \to 0$, we see that A lies on the boundary of $U_\mathfrak{m}$; in particular $A \neq 0$.

Let p, q be any integers $(q > 0)$. Then we can write $pr_k = qs_k + t_k$ where s_k, t_k are integers and $0 \leqslant t_k < q$. Then $\lim\, (t_k/q)\, A_k = 0$, so

$$\exp \frac{p}{q} A = \lim_k \exp \frac{pr_k}{q} A_k = \lim_k\, (\exp A_k)^{s_k},$$

which belongs to H. By continuity, $\exp tA \in H$ for each $t \in \boldsymbol{R}$, so $A \in \mathfrak{h}$. This contradicts the fact that $A \neq 0$ and $A \in \mathfrak{m}$.

We have therefore proved: (1) H_0 is open in H (taking $N = H^*$); (2) H_0 (and therefore H) has an analytic structure compatible with the relative topology of G in which it is a submanifold of G, hence a Lie subgroup of G. The uniqueness statement of Theorem 2.3 is immediate from Cor. 2.2.

Remark. The subgroup H above is discrete if and only if $\mathfrak{h} = 0$.

Lemma 2.5. *Let G be a Lie group and H a Lie subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the corresponding Lie algebras. Suppose H is a topological subgroup of G. Then there exists an open neighborhood V of 0 in $\mathfrak{g}$ such that:*

(i) *exp maps V diffeomorphically onto an open neighborhood of e in G.*

(ii) *$\exp (V \cap \mathfrak{h}) = (\exp V) \cap H$.*

Proof. First select a neighborhood W_0 of 0 in $\mathfrak{g}$ such that exp is one-to-one on W_0. Then select an open neighborhood N_0 of 0 in $\mathfrak{h}$ such that $N_0 \subset W_0$ and such that exp is a diffeomorphism of N_0 onto an open neighborhood N_e of e in H. Now, since H is a topological subspace of G there exists a neighborhood U_e of e in G such that $U_e \cap H = N_e$. Finally select an open neighborhood V of 0 in $\mathfrak{g}$ such that $V \subset W_0$, $V \cap \mathfrak{h} \subset N_0$ and such that exp is a diffeomorphism of V onto an open subset of G contained in U_e. Then V satisfies (i). Condition (ii) is also satisfied. In fact, let $X \in V$ such that $\exp X \in H$. Since $\exp X \in U_e \cap H = N_e$ there exists a vector $X_{\mathfrak{h}} \in N_0$ such that $\exp X_{\mathfrak{h}} = \exp X$. Since $X, X_{\mathfrak{h}} \in W_0$ we have $X = X_{\mathfrak{h}}$ so $\exp X \in \exp (V \cap \mathfrak{h})$. This proves $(\exp V) \cap H \subset \exp (V \cap \mathfrak{h})$. The converse inclusion being obvious the lemma is proved.

Theorem 2.6. *Let G and H be Lie groups and φ a continuous homomorphism of G into H. Then φ is analytic.*

Proof. Let the Lie algebras of G and H be denoted by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. The product manifold $G \times H$ is a Lie group whose Lie algebra is the product $\mathfrak{g} \times \mathfrak{h}$ as defined in §1, No. 1. The graph of φ is the subset of $G \times H$ given by $K = \{(g, \varphi(g)) : g \in G\}$. It is obvious that K is a closed subgroup of $G \times H$. As a result of Theorem 2.3, K has a unique analytic structure under which it is a topological Lie subgroup of $G \times H$. Its Lie algebra is given by

$$\mathfrak{k} = \{(X, Y) \in \mathfrak{g} \times \mathfrak{h} : (\exp tX, \exp tY) \in K \text{ for } t \in \boldsymbol{R}\}. \tag{4}$$

Let N_0 be an open neighborhood of 0 in $\mathfrak{h}$ such that exp maps N_0 diffeomorphically onto an open neighborhood N_e of e in H. Let M_0 and M_e be chosen similarly for G. We may assume that $\varphi(M_e) \subset N_e$. In view of Lemma 2.5 we can also assume that exp is a diffeomorphism of $(M_0 \times N_0) \cap \mathfrak{k}$ onto $(M_e \times N_e) \cap K$. We shall now show that for a given $X \in \mathfrak{g}$ there exists a unique $Y \in \mathfrak{h}$ such that $(X, Y) \in \mathfrak{k}$. The uniqueness is obvious from (4); in fact, if (X, Y_1) and (X, Y_2) belong to $\mathfrak{k}$, then $(0, Y_1 - Y_2) \in \mathfrak{k}$ so by (4), $(e, \exp t(Y_1 - Y_2)) \in K$ for all $t \in \boldsymbol{R}$. By the definition of K, $\exp t(Y_1 - Y_2) = \varphi(e) = e$ for $t \in \boldsymbol{R}$ so $Y_1 - Y_2 = 0$. In order to prove the existence of Y, select an integer $r > 0$ such that the vector $X_r = (1/r)\, X$ lies in M_0. Since $\varphi(\exp X_r) \in N_e$, there exists a unique vector $Y_r \in N_0$ such that $\exp Y_r = \varphi(\exp X_r)$ and a unique $Z_r \in (M_0 \times N_0) \cap \mathfrak{k}$ such that

$$\exp Z_r = (\exp X_r, \exp Y_r).$$

Now exp is one-to-one on $M_0 \times N_0$ so this relation implies $Z_r = (X_r, Y_r)$

and we can put $Y = rY_r$. The mapping $\psi : X \to Y$ thus obtained is clearly a homomorphism of $\mathfrak{g}$ into $\mathfrak{h}$. Relation (4) shows that

$$\varphi(\exp tX) = \exp t\psi(X), \qquad X \in \mathfrak{g}. \tag{5}$$

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. Then by (5)

$$\varphi((\exp t_1X_1)(\exp t_2X_2)\ldots(\exp t_nX_n))$$
$$= (\exp t_1\psi(X_1))(\exp t_2\psi(X_2))\ldots(\exp t_n\psi(X_n)). \tag{6}$$

The remark following Lemma 2.4 shows that the mapping $(\exp t_1X_1)\ldots(\exp t_nX_n) \to (t_1, \ldots, t_n)$ is a coordinate system on a neighborhood of e in G. But then by (6), φ is analytic at e, hence everywhere on G.

We shall now see that a simple countability assumption makes it possible to sharpen relation (2) and Cor. 2.2 substantially.

Proposition 2.7. *Let G be a Lie group and H a Lie subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the corresponding Lie algebras. Assume that the Lie group H has at most countably many components. Then*

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp tX \in H \text{ for all } t \in \boldsymbol{R}\}.$$

Proof. We use Lemma 2.4 for $\mathfrak{n} = \mathfrak{h}$ and $\mathfrak{m}$ any complementary subspace to $\mathfrak{h}$ in $\mathfrak{g}$. Let V denote the set $\exp U_{\mathfrak{m}} \exp U_{\mathfrak{h}}$ (from Lemma 2.4) with the relative topology of G and put

$$\mathfrak{a} = \{A \in U_{\mathfrak{m}} : \exp A \in H\}.$$

Then

$$H \cap V = \bigcup_{A \in \mathfrak{a}} \exp A \exp U_{\mathfrak{h}}$$

and this is a disjoint union due to Lemma 2.4. Each member of this union is a neighborhood in H. Since H has a countable basis the set $\mathfrak{a}$ must be countable. Consider now the mapping π of V onto $U_{\mathfrak{m}}$ given by $\pi(\exp X \exp Y) = X$ $(X \in U_{\mathfrak{m}}, Y \in U_{\mathfrak{h}})$. This mapping is continuous and maps $H \cap V$ onto $\mathfrak{a}$. The component of e in $H \cap V$ (in the topology of V) is mapped by π onto a connected countable subset of $U_{\mathfrak{m}}$, hence the single point 0. Since $\pi^{-1}(0) = \exp U_{\mathfrak{h}}$ we conclude that $\exp U_{\mathfrak{h}}$ is the component of e in $H \cap V$ (in the topology of V).

Now let $X \in \mathfrak{g}$ such that $\exp tX \in H$ for all $t \in \boldsymbol{R}$. The mapping $\varphi : t \to \exp tX$ of $\boldsymbol{R}$ into G is continuous. Hence there exists a connected neighborhood U of 0 in $\boldsymbol{R}$ such that $\varphi(U) \subset V$. Then $\varphi(U) \subset H \cap V$ and since $\varphi(U)$ is connected, $\varphi(U) \subset \exp U_{\mathfrak{h}}$. But $\exp U_{\mathfrak{h}}$ is an arbitrarily small neighborhood of e in H so the mapping φ is a continuous mapping of $\boldsymbol{R}$ into H. By (2) we have $X \in \mathfrak{h}$ and the proposition is proved.

Remark. The countability assumption is essential in Prop. 2.7 as is easily seen by considering a Lie subgroup with the discrete topology.

Corollary 2.8. *Let G be a Lie group and let H_1 and H_2 be two Lie subgroups each having countably many components. Suppose that $H_1 = H_2$ (set theoretically). Then $H_1 = H_2$ (as Lie groups).*

In fact, Prop. 2.7 shows that H_1 and H_2 have the same Lie algebra.

Corollary 2.9. *Let G be a Lie group and let K and H be two analytic subgroups of G. Assume $K \subset H$. Then the Lie group K is an analytic subgroup of the Lie group H.*

Let $\mathfrak{k}$ and $\mathfrak{h}$ denote the Lie algebras of K and H. Then $\mathfrak{k} \subset \mathfrak{h}$ by Prop. 2.7. Let K^* denote the analytic subgroup of H with Lie algebra $\mathfrak{k}$. Then the analytic subgroups K and K^* of G have the same Lie algebra. By Theorem 2.1 the Lie groups K and K^* coincide.

Let S^1 denote the unit circle and T the group $S^1 \times S^1$. Let $t \to \gamma(t)$ ($t \in R$) be a continuous one-to-one homomorphism of R into T. If we carry the analytic structure of R over by the homomorphism we obtain a Lie subgroup $\Gamma = \gamma(R)$ of T. This Lie subgroup is neither closed in T nor a topological subgroup of T. We shall now see that these anomalies go together.

Theorem 2.10. *Let G be a Lie group and H a Lie subgroup of G.*

(i) *If H is a topological subgroup of G then H is closed in G.*

(ii) *If H has at most countably many components and is closed in G then H is a topological subgroup of G.*

Part (i) is contained in a more general result.

Proposition 2.11. *Let G be a topological group and H a subgroup which in the relative topology is locally compact. Then H is closed in G. In particular, if H, in the relative topology, is discrete, then it is closed.*

Proof. We first construct, without using the group structure, an open set $V \subset G$ whose intersection with the closure $\bar{H}$ is H. Let $h \in H$, let U_h be a compact neighborhood of h in H, and V_h a neighborhood of h in G such that $V_h \cap H = U_h$. Let $\mathring{V}_h$ be the interior of V_h. If $g \in \mathring{V}_h \cap \bar{H}$ and N_g is any neighborhood of g in G, then N_g intersects the closed set $V_h \cap H$ $(N_g \cap (V_h \cap H) \supset (N_g \cap \mathring{V}_h) \cap H \neq \emptyset)$, whence $g \in V_h \cap H$. Thus $\mathring{V}_h \cap \bar{H} \subset V_h \cap H$ so $\mathring{V}_h \cap \bar{H} = \mathring{V}_h \cap H$. Taking $V = \bigcup_{h \in H} \mathring{V}_h$, we have $H = \bar{H} \cap V$.

Let $b \in H$ and W be a neighborhood of e in G such that $bW \subset V$. If $a \in \bar{H}$, $aW^{-1} \cap H$ contains an element c; hence $bc^{-1}a \subset bW \subset V$. Also $bc^{-1} \in H$, so $bc^{-1}a \in \bar{H}$. Thus $bc^{-1}a \in V \cap \bar{H} = H$, so $a \in H$. Q.E.D.

(ii) H being closed in G it has by Theorem 2.3 an analytic structure in which it is a topological Lie subgroup of G. Let H' denote this Lie subgroup. Then the identity mapping $I : H \to H'$ is continuous. Each component of H lies in a component of H'. Since H has countably many components the same holds for H'. Now (ii) follows from Cor. 2.8.

§ 3. Lie Transformation Groups

Let M be a Hausdorff space and G a topological group such that to each $g \in G$ is associated a homeomorphism $p \to g \cdot p$ of M onto itself such that

(1) $g_1 g_2 \cdot p = g_1 \cdot (g_2 \cdot p)$ for $p \in M$, $g_1, g_2 \in G$;

(2) the mapping $(g, p) \to g \cdot p$ is a continuous mapping of the product space $G \times M$ onto M.

The group G is then called a *topological transformation group* of M. From (1) follows that $e \cdot p = p$ for all $p \in M$. If e is the only element of G which leaves each $p \in M$ fixed then G is called *effective* and is said to act effectively on M.

Example. Suppose A is a topological group and F a closed subgroup of A. The system of left cosets aF, $a \in A$ is denoted A/F; let π denote the natural mapping of A onto A/F. The set A/F can be given a topology, the *natural topology*, which is uniquely determined by the condition that π is a continuous and open mapping. This makes A/F a Hausdorff space and it is not difficult to see that if to each $a \in A$ we assign the mapping $\tau(a) : bF \to abF$, then A is a topological transformation group of A/F. The group A is effective if and only if F contains no normal subgroup of A. The coset space A/F is a *homogeneous space*, that is, has a transitive group of homeomorphisms, namely $\tau(A)$. Theorem 3.2 below deals with the converse question, namely that of representing a homogeneous space by means of a coset space.

Lemma 3.1 (the category theorem). *If a locally compact space M is a countable union*

$$M = \bigcup_{n=1}^{\infty} M_n,$$

where each M_n is a closed subset, then at least one M_n contains an open subset of M.

Proof. Suppose no M_n contains an open subset of M. Let U_1 be an open subset of M whose closure $\bar{U}_1$ is compact. Select successively

$a_1 \in U_1 - M_1$ and a neighborhood U_2 of a_1 such that $\bar{U}_2 \subset U_1$ and $\bar{U}_2 \cap M_1 = \emptyset$;

$a_2 \in U_2 - M_2$ and a neighborhood U_3 of a_2 such that $\bar{U}_3 \subset U_2$ and $\bar{U}_3 \cap M_2 = \emptyset$, etc.

Then $\bar{U}_1, \bar{U}_2, \ldots$ is a decreasing sequence of compact sets $\neq \emptyset$. Thus there is a point $b \in M$ in common to all $\bar{U}_n$. But this implies $b \notin M_n$ for each n which is a contradiction.

Theorem 3.2. *Let G be a locally compact group with a countable base. Suppose G is a transitive topological transformation group of a locally compact Hausdorff space M. Let p be any point in M and H the subgroup of G which leaves p fixed. Then H is closed and the mapping*

$$gH \to g \cdot p$$

is a homeomorphism of G/H onto M.

Proof. Since the mapping $\varphi : g \to g \cdot p$ of G onto M is continuous, it follows that $H = \varphi^{-1}(p)$ is closed in G. The natural mapping $\pi : G \to G/H$ is open and continuous. Thus, in order to prove Theorem 3.2, it suffices to prove that φ is open. Let V be an open subset of G and g a point in V. Select a compact neighborhood U of e in G such that $U = U^{-1}$, $gU^2 \subset V$. There exists a sequence $(g_n) \subset G$ such that $G = \bigcup_n g_n U$. The group G being transitive, this implies $M = \bigcup_n g_n U \cdot p$. Each summand is compact, hence a closed subset of M. By the lemma above, some summand, and therefore $U \cdot p$, contains an inner point $u \cdot p$. Then p is an inner point of $u^{-1}U \cdot p \subset U^2 \cdot p$ and consequently $g \cdot p$ is an inner point of $V \cdot p$. This shows that the mapping φ is open.

Definition. The group H is called the *isotropy* group at p (or the isotropy subgroup of G at p).

Corollary 3.3. *Let G and X be two locally compact groups. Assume G has a countable base. Then every continuous homomorphism ψ of G onto X is open.*

In fact, if we associate to each $g \in G$ the homeomorphism $x \to \psi(g)\, x$ of X onto itself, then G becomes a transitive topological transformation group of X. If f denotes the identity element of X, the proof above shows that the mapping $g \to \psi(g)f$ of G onto X is open.

Let G be a Lie group and M a differentiable manifold. Suppose G is a topological transformation group of M; G is said to be a *Lie transformation group* of M if the mapping $(g, p) \to g \cdot p$ is a differentiable mapping of $G \times M$ onto M. It follows that for each $g \in G$ the mapping $p \to g \cdot p$ is a diffeomorphism of M onto itself.

Let G be a Lie transformation group of M. To each X in $\mathfrak{g}$, the Lie algebra of G, we can associate a vector field X^+ on M by the formula

$$[X^+f](p) = \lim_{t\to 0} \frac{f(\exp tX \cdot p) - f(p)}{t}$$

for $f \in C^\infty(M)$, $p \in M$. The existence of X^+ follows from the fact that the mapping $(g, p) \to g \cdot p$ is a differentiable mapping of $G \times M$ onto M. It is also easy to check that X^+ is a derivation of $C^\infty(M)$. It is called the vector field on M *induced* by the one-parameter subgroup $\exp tX$, $t \in \boldsymbol{R}$.

Theorem 3.4. *Let G be a Lie transformation group of M. Let X, Y be in $\mathfrak{g}$, the Lie algebra of G, and let X^+, Y^+ be the vector fields on M induced by $\exp tX$ and $\exp tY$ ($t \in \boldsymbol{R}$). Then*

$$[X^+, Y^+] = -[X, Y]^+. \tag{1}$$

We first prove a lemma which also shows what would have happened had we used right translation $R_p : g \to gp$ instead of left translation in the definition of the Lie algebra.

Lemma 3.5. *Let $\tilde{X}$ and $\tilde{Y}$ denote the right invariant vector fields on G such that $\tilde{X}_e = X$, $\tilde{Y}_e = Y$. Then*

$$[\tilde{X}, \tilde{Y}] = -[X, Y]^{\sim}. \tag{2}$$

Proof. In analogy with (5), §1 we have

$$(\tilde{X}f)(g) = \left\{\frac{d}{dt} f(\exp(tX)g)\right\}_{t=0} \tag{3}$$

for $f \in C^\infty(G)$. Then if J denotes the diffeomorphism $g \to g^{-1}$ of G,

$$dJ_g(\tilde{X}_g)f = \left\{\frac{d}{dt}(f \circ J)(g \exp tX)\right\}_{t=0} = -X(f \circ R_{g^{-1}}) = -\tilde{X}_{g^{-1}}f.$$

Thus $dJ(\tilde{X}) = -\tilde{X}$, so (2) follows from Prop. 3.3, Chapter I.

For (1) fix $p \in M$ and the map $\Phi : g \in G \to g \cdot p \in M$. Then

$$d\Phi_g(\tilde{X}_g)f = \tilde{X}_g(f \circ \Phi) = \left\{\frac{d}{dt} f(\exp(tX)g \cdot p)\right\}_{t=0}.$$

Thus $d\Phi_g(\overline{X}_g) = X^+_{\Phi(g)}$, so $d\Phi_g([\overline{X}, \overline{Y}]_g) = [X^+, Y^+]_{\Phi(g)}$. Using (2) we deduce

$$[X^+, Y^+]_{\Phi(g)} = -[X, Y]^+_{\Phi(g)}$$

so since p is arbitrary, (1) follows.

We now come back to the introductory material in the first lecture and prove Lie's theorem stated there.

First we explain how Theorem 3.4 is connected with the original foundation of Lie group theory. Inspired by Galois' theory for algebraic equations, Lie raised the following question in his paper [2]: *How can the knowledge of a stability group for a differential equation be utilized toward its integration* ? (A point transformation is said to leave a differential equation *stable* if it permutes the solutions.) Lie proved in [2] that a one-parameter transformation group φ_t of R^2 with induced vector field

$$\Phi_p = \left(\frac{d(\varphi_t \cdot p)}{dt}\right)_{t=0} = \xi(p)\frac{\partial}{\partial x} + \eta(p)\frac{\partial}{\partial y}$$

leaves a differential equation $dy/dx = Y(x, y)/X(x, y)$ stable if and only if the vector field $Z = X\,\partial/\partial x + Y\,\partial/\partial y$ satisfies $[\Phi, Z] = \lambda Z$ where λ is a function; in this case $(X\eta - Y\xi)^{-1}$ is an integrating factor for the equation $X\,dy - Y\,dx = 0$.

Example.

$$\frac{dy}{dx} = \frac{y + x(x^2+y^2)}{x - y(x^2+y^2)}.$$

This equation can be written

$$\left(\frac{dy}{dx} - \frac{y}{x}\right)\Big/\left(1 + \frac{y}{x}\frac{dy}{dx}\right) = x^2 + y^2;$$

and since the left-hand side is $\tan\alpha$ where α is the angle between the integral curve through (x, y) and the radius vector, it is clear that the integral curves intersect each circle around $(0, 0)$ under a fixed angle. Thus the rotation group

$$\varphi_t : (x, y) \to (x\cos t - y\sin t,\ x\sin t + y\cos t),$$

for which $\Phi = -y\,\partial/\partial x + x\,\partial/\partial y$, leaves the equation stable and Lie's theorem gives the solution $y = x\tan(\frac{1}{2}(x^2+y^2) + C)$, C a constant.

Generalizing (φ_t) above, Lie considered transformations $(x_1, ..., x_n) \to (x_1', ..., x_n')$ given by

$$T : x_i' = f_i(x_1, ..., x_n; t_1, ..., t_r) \tag{1}$$

depending effectively on r parameters t_k, i.e., the f_i are C^∞ functions and the matrix $(\partial f_i/\partial t_k)$ has rank r. We assume that the identity transformation is given by $t_1 = \cdots = t_r = 0$ and that if a transformation S corresponds to the parameters $(s_1, ..., s_r)$ then TS^{-1} is for sufficiently small t_i, s_j given by

$$TS^{-1} : x_i' = f_i(x_1, ..., x_n; u_1, ..., u_r) \tag{2}$$

where the u_k are analytic functions of the t_i and s_j. Generalizing Φ above, Lie introduced the vector fields

$$T_k = \sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial t_k}\right)_{t=0}\frac{\partial}{\partial x_i} \qquad (1 \leqslant k \leqslant r)$$

and, as a result of the group property (2), proved the fundamental formula

$$[T_k, T_l] = \sum_{p=1}^{r} c^p{}_{kl} T_p, \tag{3}$$

where the $c^p{}_{kl}$ are *constants* satisfying

$$c^p{}_{kl} = -c^p{}_{lk}, \qquad \sum_{q=1}^{r} (c^l{}_{kq} c^q{}_{lm} + c^p{}_{mq} c^q{}_{kl} + c^p{}_{lq} c^q{}_{mk}) = 0. \tag{4}$$

Independently of Lie, Killing had through geometric investigations been led to concepts close to T_k and relations (3) and he attacked the algebraic problem of classifying all solutions to (4). See **Additional Readings** 3.

We shall now show how (3) follows from Theorem 3.4. So the (x_i) are coordinates on M and the (t_k) coordinates near e in G. We first assume that thc (t_k) are "canonical coordinates of the second kind," i.e., for a suitable basis $X_1, \ldots, X_r$ of $\mathfrak{g}$,

$$\begin{aligned} &\exp(t_1 X_1) \ldots \exp(t_r X_r) \cdot (x_1, \ldots, x_n) \\ &\quad = f_1(x_1, \ldots, x_n; t_1, \ldots, t_r), \ldots, f_n(x_1, \ldots, x_n; t_1, \ldots, t_r). \end{aligned}$$

Such coordinates exist by the remark following Lemma 2.4. Then $X_k = (\partial/\partial t_k)_e$ $(1 \leqslant k \leqslant r)$ and

$$(X_k^+ f)(p) = \sum_{i=1}^{n} (X_k^+ x_i)(p) \left(\frac{\partial}{\partial x_i}\right)_p (f),$$

$$\begin{aligned} (X_k^+ x_i)(p) &= \left\{\frac{d}{ds} x_i(\exp(sX_k) \cdot (x_1, \ldots, x_n))\right\}_{s=0} \\ &= \left\{\frac{\partial}{\partial s} f_i(x_1, \ldots, x_n; 0, \ldots, s, \ldots, 0)\right\}_{s=0} = \left(\frac{\partial f_i}{\partial t_k}\right)_{t=0}. \end{aligned}$$

It follows that $X_+^k = T_k$, so Theorem 3.4 implies (3) for this coordinate system $\{t_1, \ldots, t_r\}$. But if $\{s_1, \ldots, s_r\}$ is another coordinate system on a neighborhood of e in G, with $s_1(e) = \cdots = s_r(e) = 0$, then

$$f_i(x_1, \ldots, x_n; s_1, \ldots, s_r) = f_i'(x_1, \ldots, x_n; t_1, \ldots, t_r)$$

where the f_i' are obtained by changing to the canonical coordinates $(t_1, \ldots, t_k)$ of the second kind. Then

$$\frac{\partial f_i}{\partial s_k} = \sum_{l=1}^{r} \frac{\partial f'_i}{\partial t_l} \frac{\partial t_l}{\partial s_k}$$

so the new T_k are certain constant linear combinations of the old T_ℓ; thus (3) holds in general.

§ 4. Coset Spaces and Homogeneous Spaces

Let G be a Lie group and H a closed subgroup. The group H will always be given the analytic structure from Theorem 2.3. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively, and let $\mathfrak{m}$ denote some vector subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (direct sum). Let π be the natural mapping of G onto the space G/H of left cosets gH, $g \in G$. As usual we give G/H the natural topology determined by the requirement that π should be continuous and open. We put $p_0 = \pi(e)$ and let ψ denote the restriction of exp to $\mathfrak{m}$.

Lemma 4.1. *There exists a neighborhood U of 0 in $\mathfrak{m}$ which is mapped homeomorphically under ψ and such that π maps $\psi(U)$ homeomorphically onto a neighborhood of p_0 in G/H.*

Proof. Let $U_{\mathfrak{m}}$, $U_{\mathfrak{h}}$ have the property described in Lemma 2.4 for $\mathfrak{h} = \mathfrak{n}$. Then since H has the relative topology of G, we can select a neighborhood V of e in G such that $V \cap H = \exp U_{\mathfrak{h}}$. Let U be a compact neighborhood of 0 in $U_{\mathfrak{m}}$ such that $\exp(-U) \exp U \subset V$. Then ψ is a homeomorphism of U onto $\psi(U)$. Moreover, π is one-to-one on $\psi(U)$ because if $X', X'' \in U$ satisfy $\pi(\exp X') = \pi(\exp X'')$, then $\exp(-X'') \exp X' \subset V \cap H$ so $\exp X' = \exp X'' \exp Z$ where $Z \in U_{\mathfrak{h}}$. From Lemma 2.4 we can conclude that $X' = X''$, $Z = 0$; consequently, π is one-to-one on $\psi(U)$, hence a homeomorphism. Finally, $U \times U_{\mathfrak{h}}$ is a neighborhood of $(0, 0)$ in $U_{\mathfrak{m}} \times U_{\mathfrak{h}}$; hence $\exp U \exp U_{\mathfrak{h}}$ is a neighborhood of e in G and since π is an open mapping, the set $\pi(\exp U \exp U_{\mathfrak{h}}) = \pi(\psi(U))$ is a neighborhood of p_0 in G/H. This proves the lemma. The set $\psi(U)$ will be referred to as a *local cross section.*

Let N_0 denote the interior of the set $\pi(\psi(U))$ and let $X_1, \ldots, X_r$ be a basis of $\mathfrak{m}$. If $g \in G$, then the mapping

$$\pi(g \exp(x_1 X_1 + \ldots + x_r X_r)) \rightarrow (x_1, \ldots, x_r)$$

is a homeomorphism of the open set $g \cdot N_0$ onto an open subset of $\boldsymbol{R}^r$. It is easy to see[†] that with these charts, G/H is an analytic manifold. Moreover, if $x \in G$, the mapping $\tau(x) : yH \rightarrow xyH$ is an analytic diffeomorphism of G/H.

Theorem 4.2. *Let G be a Lie group, H a closed subgroup of G, G/H the space of left cosets gH with the natural topology. Then G/H has a unique analytic structure with the property that G is a Lie transformation group of G/H.*

We use the notation above and let $B = \psi(\hat{U})$ where $\hat{U}$ is the interior of U. Remembering that the mapping Φ in Lemma 2.4 is a diffeo-

[†] See Exercise C.4.

morphism, the set B is a submanifold of G. The mappings in the diagram

$$\begin{array}{ccc} G \times B & \xrightarrow{\Phi} & G \\ {\scriptstyle I \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times N_0 & & G/H \end{array} \quad \text{are:} \quad \begin{array}{ll} I \times \pi : (g, x) \to (g, xH), & g \in G, x \in B; \\ \Phi : (g, x) \to gx, & g \in G, x \in B. \end{array}$$

Then the mapping $(g, xH) \to gxH$ of $G \times N_0$ onto G/H can be written $\pi \circ \Phi \circ (I \times \pi)^{-1}$ which is analytic. Thus G is a Lie transformation group of G/H. The uniqueness results from the following proposition which should be compared with Theorem 3.2.

Proposition 4.3. *Let G be a transitive Lie transformation group of a C^∞ manifold M. Let p_0 be a point in M and let G_{p_0} denote the subgroup of G that leaves p_0 fixed. Then G_{p_0} is closed. Let α denote the mapping $gG_{p_0} \to g \cdot p_0$ of G/G_{p_0} onto M.*

(a) *If α is a homeomorphism, then it is a diffeomorphism (G/G_{p_0} having the analytic structure defined above).*

(b) *Suppose α is a homeomorphism and that M is connected. Then G_0, the identity component of G, is transitive on M.*

Proof. (a) We put $H = G_{p_0}$ and use Lemma 4.1. Let B and N_0 have the same meaning as above. Then B is a submanifold of G, diffeomorphic to N_0 under π. Let i denote the identity mapping of B into G and let β denote the mapping $g \to g \cdot p_0$ of G onto M. By assumption, α_{N_0}, the restriction of α to N_0, is a homeomorphism of N_0 onto an open subset of M. The mapping α is differentiable since $\alpha_{N_0} = \beta \circ i \circ \pi^{-1}$. To show that α^{-1} is differentiable, we begin by showing that the Jacobian of β at $g = e$ has rank r_β equal to dim M.

$$\begin{array}{ccc} B & \xrightarrow{i} & G \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \beta} \\ N_0 & \xrightarrow{\alpha} & M \end{array}$$

The mapping $d\beta_e$ is a linear mapping of $\mathfrak{g}$ into M_{p_0}. Suppose X is in the kernel of $d\beta_e$. Then if $f \in C^\infty(M)$, we have

$$0 = (d\beta_e X)f = X(f \circ \beta) = \left\{\frac{d}{dt} f(\exp tX \cdot p_0)\right\}_{t=0}. \qquad (1)$$

Let $s \in \boldsymbol{R}$; we use (1) on the function $f^*(q) = f(\exp sX \cdot q)$, $q \in M$. Then

$$0 = \left\{\frac{d}{dt} f^*(\exp tX \cdot p_0)\right\}_{t=0} = \left\{\frac{d}{dt} f(\exp tX \cdot p_0)\right\}_{t=s},$$

which shows that $f(\exp sX \cdot p_0)$ is constant in s. Since f is arbitrary, we have $\exp sX \cdot p_0 = p_0$ for all s so $X \in \mathfrak{h}$. On the other hand, it is obvious that $d\beta_e$ vanishes on $\mathfrak{h}$ so $\mathfrak{h} = \text{kernel } (d\beta_e)$. Hence $r_\beta = \dim \mathfrak{g} - \dim \mathfrak{h}$. But α is a homeomorphism, so the topological invariance of dimension implies $\dim G/H = \dim M$. Thus $r_\beta = \dim M$, α^{-1} is differentiable at p and, by translation, on M. This proves (a).

(b) If α is a homeomorphism, β above is an open mapping. There exists a subset $\{x_\gamma : \gamma \in I\}$ of G such that $G = \bigcup_{\gamma \in I} G_0 x_\gamma$. Each orbit $G_0 x_\gamma \cdot p_0$ is an open subset of M; two orbits $G_0 x_\gamma \cdot p_0$ and $G_0 x_{\gamma'} \cdot p_0$ are either disjoint or equal. Therefore, since M is connected, all orbits must coincide and (b) follows.

Definition. In the sequel the coset space G/H (G a Lie group, H a closed subgroup) will always be taken with the analytic structure described in Theorem 4.2. If $x \in G$, the diffeomorphism $yH \to xyH$ of G/H onto itself will be denoted by $\tau(x)$. The group H is called the *isotropy group*. The group H^* of linear transformations $(d\tau(h))_{\pi(e)}$, $(h \in H)$, is called the *linear isotropy group*.

§ 5. The Adjoint Group

Let $\mathfrak{a}$ be a Lie algebra over $\boldsymbol{R}$. Let $\boldsymbol{GL}(\mathfrak{a})$ as usual denote the group of all nonsingular endomorphisms of $\mathfrak{a}$. We recall that an endomorphism of a vector space V (in particular of a Lie algebra) simply means a linear mapping of V into itself. The Lie algebra $\mathfrak{gl}(\mathfrak{a})$ of $\boldsymbol{GL}(\mathfrak{a})$ consists of the vector space of all endomorphisms of $\mathfrak{a}$ with the bracket operation $[A, B] = AB - BA$. The mapping $X \to \text{ad } X$, $X \in \mathfrak{a}$ is a homomorphism of $\mathfrak{a}$ onto a subalgebra ad $(\mathfrak{a})$ of $\mathfrak{gl}(\mathfrak{a})$. Let Int $(\mathfrak{a})$ denote the analytic subgroup of $\boldsymbol{GL}(\mathfrak{a})$ whose Lie algebra is ad $(\mathfrak{a})$; Int $(\mathfrak{a})$ is called the *adjoint group* of $\mathfrak{a}$.

The group Aut $(\mathfrak{a})$ of all automorphisms of $\mathfrak{a}$ is a closed subgroup of $\boldsymbol{GL}(\mathfrak{a})$. Thus Aut $(\mathfrak{a})$ has a unique analytic structure in which it is a topological Lie subgroup of $\boldsymbol{GL}(\mathfrak{a})$. Let $\partial(\mathfrak{a})$ denote the Lie algebra of Aut $(\mathfrak{a})$. From §2 we know that $\partial(\mathfrak{a})$ consists of all endomorphisms D of $\mathfrak{a}$ such that $e^{tD} \in \text{Aut}(\mathfrak{a})$ for each $t \in \boldsymbol{R}$. Let $X, Y \in \mathfrak{a}$. The relation $e^{tD}[X, Y] = [e^{tD}X, e^{tD}Y]$ for all $t \in \boldsymbol{R}$ implies

$$D[X, Y] = [DX, Y] + [X, DY]. \tag{1}$$

An endomorphism D of $\mathfrak{a}$ satisfying (1) for all $X, Y \in \mathfrak{a}$ is called a *derivation* of $\mathfrak{a}$. By induction we get from (1)

$$D^k[X, Y] = \sum_{i+j=k} \frac{k!}{i!j!} [D^iX, D^jY], \qquad i \geqslant 0, j \geqslant 0, \tag{2}$$

where D^0 means the identity mapping of $\mathfrak{a}$. From (2) follows that $e^{tD}[X, Y] = [e^{tD}X, e^{D}Y]$ and thus $\partial(\mathfrak{a})$ *consists of all derivations of* $\mathfrak{a}$. Using the Jacobi identity we see that $\text{ad}(\mathfrak{a}) \subset \partial(\mathfrak{a})$ and therefore $\text{Int}(\mathfrak{a}) \subset \text{Aut}(\mathfrak{a})$. The elements of $\text{ad}(\mathfrak{a})$ and $\text{Int}(\mathfrak{a})$, respectively, are called the *inner derivations* and *inner automorphisms* of $\mathfrak{a}$. Since $\text{Aut}(\mathfrak{a})$ is a topological subgroup of $\boldsymbol{GL}(\mathfrak{a})$ the identity mapping of $\text{Int}(\mathfrak{a})$ into $\text{Aut}(\mathfrak{a})$ is continuous. In view of Lemma 14.1, Chapter I, $\text{Int}(\mathfrak{a})$ is a Lie subgroup of $\text{Aut}(\mathfrak{a})$. We shall now prove that $\text{Int}(\mathfrak{a})$ is a normal subgroup of $\text{Aut}(\mathfrak{a})$. Let $s \in \text{Aut}(\mathfrak{a})$. Then the mapping $\sigma : g \to sgs^{-1}$ is an automorphism of $\text{Aut}(\mathfrak{a})$, and $(d\sigma)_e$ is an automorphism of $\partial(\mathfrak{a})$. If A, B are endomorphisms of a vector space and A^{-1} exists, then $Ae^{B}A^{-1} = e^{ABA^{-1}}$. Considering Lemma 1.12 we have

$$(d\sigma)_e D = sDs^{-1} \qquad \text{for } D \in \partial(\mathfrak{a}).$$

If $X \in \mathfrak{a}$, we have $s \,\text{ad}\, X s^{-1} = \text{ad}(s \cdot X)$, so

$$(d\sigma)_e \,\text{ad}\, X = \text{ad}(s \cdot X),$$

and consequently

$$\sigma \cdot e^{\text{ad}\, X} = e^{\text{ad}(s \cdot X)} \qquad (X \in \mathfrak{a}).$$

Now, the group $\text{Int}(\mathfrak{a})$ is connected, so it is generated by the elements $e^{\text{ad}\, X}$, $X \in \mathfrak{a}$. It follows that $\text{Int}(\mathfrak{a})$ is a normal subgroup of $\text{Aut}(\mathfrak{a})$ and the automorphism s of $\mathfrak{a}$ induces the analytic isomorphism $g \to sgs^{-1}$ of $\text{Int}(\mathfrak{a})$ onto itself.

More generally, if s is an isomorphism of a Lie algebra $\mathfrak{a}$ onto a Lie algebra $\mathfrak{b}$ (both Lie algebras over $\boldsymbol{R}$) then the mapping $g \to sgs^{-1}$ is an isomorphism of $\text{Aut}(\mathfrak{a})$ onto $\text{Aut}(\mathfrak{b})$ which maps $\text{Int}(\mathfrak{a})$ onto $\text{Int}(\mathfrak{b})$.

Let G be a Lie group. If $\sigma \in G$, the mapping $I(\sigma) : g \to \sigma g \sigma^{-1}$ is an analytic isomorphism of G onto itself. We put $\text{Ad}(\sigma) = dI(\sigma)_e$. Sometimes we write $\text{Ad}_G(\sigma)$ instead of $\text{Ad}(\sigma)$ when a misunderstanding might otherwise arise. The mapping $\text{Ad}(\sigma)$ is an automorphism of $\mathfrak{g}$, the Lie algebra of G. We have by Lemma 1.12

$$\exp \text{Ad}(\sigma) X = \sigma \exp X \sigma^{-1} \qquad \text{for } \sigma \in G,\ X \in \mathfrak{g}. \tag{3}$$

The mapping $\sigma \to \text{Ad}(\sigma)$ is a homomorphism of G into $\boldsymbol{GL}(\mathfrak{g})$. This homomorphism is called the *adjoint representation* of G. Let us prove that this homomorphism is analytic. For this it suffices to prove that for each $X \in \mathfrak{g}$ and each linear function ω on $\mathfrak{g}$ the function $\sigma \to \omega(\text{Ad}(\sigma) X)$ $(\sigma \in G)$, is analytic at $\sigma = e$. Select $f \in C^{\infty}(G)$ such that f is analytic at $\sigma = e$ and such that $Yf = \omega(Y)$ for all $Y \in \mathfrak{g}$.

Then, using (3), we obtain

$$\omega(\mathrm{Ad}(\sigma)X) = (\mathrm{Ad}(\sigma)X)f = \left\{\frac{d}{dt}f(\sigma \exp tX\sigma^{-1})\right\}_{t=0},$$

which is clearly analytic at $\sigma = e$.

Next, let X and Y be arbitrary vectors in $\mathfrak{g}$. From Lemma 1.8 (iii) we have

$$\exp(\mathrm{Ad}(\exp tX)\,tY) = \exp(tY + t^2[X, Y] + O(t^3)).$$

It follows that

$$\mathrm{Ad}(\exp tX)\,Y = Y + t[X, Y] + O(t^2). \tag{4}$$

The differential $d\,\mathrm{Ad}_e$ is a homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{g})$ and due to (4) we have

$$d\,\mathrm{Ad}_e(X) = \mathrm{ad}\,X, \qquad X \in \mathfrak{g}.$$

Applying the exponential mapping on both sides we obtain (Lemma 1.12)

$$\mathrm{Ad}(\exp X) = e^{\mathrm{ad}\,X}, \qquad X \in \mathfrak{g}. \tag{5}$$

Note that if $[X, Y] = 0$ then by (5)

$$\exp X \exp Y \exp(-X) = \exp A\,d(\exp X)Y = \exp Y$$

so

$$\exp X \exp Y = \exp Y \exp X = \exp(X + Y),$$

the last identity coming from the fact that now $t \to \exp tX \exp tY$ is a one-parameter subgroup with tangent vector $X + Y$ at e.

Let G be a connected Lie group and H an analytic subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the corresponding Lie algebras. Relations (3) and (5) show that H is a normal subgroup of G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

Lemma 5.1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let φ be an analytic homomorphism of G into a Lie group X with Lie algebra $\mathfrak{x}$. Then:*

(i) *The kernel $\varphi^{-1}(e)$ is a topological Lie subgroup of G. Its Lie algebra is the kernel of $d\varphi$ $(= d\varphi_e)$.*

(ii) *The image $\varphi(G)$ is a Lie subgroup of X with Lie algebra $d\varphi(\mathfrak{g}) \subset \mathfrak{x}$.*

(iii) *The factor group $G/\varphi^{-1}(e)$ with its natural analytic structure is a Lie group and the mapping $g\varphi^{-1}(e) \to \varphi(g)$ is an analytic isomorphism of $G/\varphi^{-1}(e)$ onto $\varphi(G)$. In particular, the mapping $\varphi : G \to \varphi(G)$ is analytic.*

Proof. (i) According to Theorem 2.3, $\varphi^{-1}(e)$ has a unique analytic structure with which it is a topological Lie subgroup of G. Moreover, its Lie algebra contains a vector $Z \in \mathfrak{g}$ if and only if $\varphi(\exp tZ) = e$ for all $t \in \boldsymbol{R}$. Since $\varphi(\exp tZ) = \exp td\varphi(Z)$, the condition is equivalent to $d\varphi(Z) = 0$.

(ii) Let X_1 denote the analytic subgroup of X with Lie algebra $d\varphi(\mathfrak{g})$. The group $\varphi(G)$ is generated by the elements $\varphi(\exp Z)$, $Z \in \mathfrak{g}$. The group X_1 is generated by the elements $\exp(d\varphi(Z))$, $Z \in \mathfrak{g}$. Since $\varphi(\exp Z) = \exp d\varphi(Z)$ it follows that $\varphi(G) = X_1$.

(iii) Let H be any closed normal subgroup of G. Then H is a topological Lie subgroup and the factor group G/H has a unique analytic structure such that the mapping $(g, xH) \to g\,xH$ is an analytic mapping of $G \times G/H$ onto G/H. In order to see that G/H is a Lie group in this analytic structure we use the local cross section $\psi(U)$ from Lemma 4.1. Let $B = \psi(\mathring{U})$ where $\mathring{U}$ is the interior of U. In the commutative diagram

$$\begin{tikzcd} G \times G/H \arrow[rr, "\Phi"] \arrow[dr, "\pi \times I"'] & & G/H \\ & G/H \times G/H \arrow[ur, "\alpha"'] & \end{tikzcd}$$

the symbols Φ, $\pi \times I$, and α denote the mappings:

$$\begin{aligned} \Phi &: (g, xH) \to g^{-1}xH, && x, g \in G; \\ \pi \times I &: (g, xH) \to (gH, xH), && x, g \in G; \\ \alpha &: (gH, xH) \to g^{-1}xH, && x, g \in G. \end{aligned}$$

The mapping α is well defined since H is a normal subgroup of G. Let g_0, x_0 be arbitrary two points in G. The restriction of $\pi \times I$ to $(g_0B) \times (G/H)$ is an analytic diffeomorphism of $(g_0B) \times (G/H)$ onto a neighborhood N of (g_0H, x_0H) in $G/H \times G/H$. On N we have $\alpha = \Phi \circ (\pi \times I)^{-1}$ which shows that α is analytic. Hence G/H is a Lie group.

Now choose for H the group $\varphi^{-1}(e)$ and let $\mathfrak{h}$ denote the Lie algebra of H. Then $\mathfrak{h} = d\varphi^{-1}(0)$ so $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. By (ii) the Lie algebra of G/H is $d\pi(\mathfrak{g})$ which is isomorphic to the algebra $\mathfrak{g}/\mathfrak{h}$. On the other hand, the mapping $Z + \mathfrak{h} \to d\varphi(Z)$ is an isomorphism of $\mathfrak{g}/\mathfrak{h}$ onto $d\varphi(\mathfrak{g})$. The corresponding local isomorphism between G/H and $\varphi(G)$ coincides with the (algebraic) isomorphism $gH \to \varphi(g)$ on some neighborhood of the identity. It follows that this last isomorphism is analytic at e, hence everywhere.

Corollary 5.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let Z denote the center of G. Then:*

(i) *Ad_G is an analytic homomorphism of G onto $Int\,(\mathfrak{g})$ with kernel Z.*

(ii) *The mapping $gZ \to Ad_G(g)$ is an analytic isomorphism of G/Z onto $Int\,(\mathfrak{g})$.*

In fact $\mathrm{Ad}_G\,(G) = \mathrm{Int}\,(\mathfrak{g})$ due to (5) and $\mathrm{Ad}_G^{-1}(e) = Z$ due to (3). The remaining statements are contained in Lemma 5.1.

Corollary 5.3. *Let $\mathfrak{g}$ be a Lie algebra over R with center $\{0\}$. Then the center of $Int\,(\mathfrak{g})$ consists of the identity element alone.*

In fact, let $G' = \operatorname{Int}(\mathfrak{g})$ and let Z denote the center of G'. Let ad denote the adjoint representation of $\mathfrak{g}$ and let Ad' and ad' denote the adjoint representation of G' and ad $(\mathfrak{g})$, respectively. The mapping

$$\theta : gZ \to \operatorname{Ad}'(g), \qquad g \in G',$$

is an isomorphism of G'/Z onto Int (ad $(\mathfrak{g})$). On the other hand, the mapping $s : X \to \operatorname{ad} X$ $(X \in \mathfrak{g})$ is an isomorphism of $\mathfrak{g}$ onto ad $(\mathfrak{g})$ and consequently the mapping $S : g \to s \circ g \circ s^{-1}$ $(g \in G')$ is an isomorphism of G' onto Int (ad $(\mathfrak{g})$). Moreover, if $X \in \mathfrak{g}$, we obtain from (5)

$$S(e^{\operatorname{ad} X}) = s \circ e^{\operatorname{ad} X} \circ s^{-1} = e^{(\operatorname{ad}'(\operatorname{ad} X))} = \operatorname{Ad}'(e^{\operatorname{ad} X}),$$

ad $(\mathfrak{g})$ being the Lie algebra of G'. It follows that $S^{-1} \circ \theta$ is an isomorphism of G'/Z onto G', mapping gZ onto g $(g \in G')$. Obviously Z must consist of the identity element alone.

Remark. The conclusion of Cor. 5.3 does not hold in general, if $\mathfrak{g}$ has nontrivial center. Let, for example, $\mathfrak{g}$ be the three-dimensional Lie algebra $\mathfrak{g} = \boldsymbol{R}X_1 + \boldsymbol{R}X_2 + \boldsymbol{R}X_3$ with the bracket defined by: $[X_1, X_2] = X_3$, $[X_1, X_3] = [X_2, X_3] = 0$. Here $\mathfrak{g}$ is nonabelian, whereas Int $(\mathfrak{g})$ is abelian and has dimension 2.

Definition. Let $\mathfrak{g}$ be a Lie algebra over $\boldsymbol{R}$. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ and K^* the analytic subgroup of Int $(\mathfrak{g})$ which corresponds to the subalgebra $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{k})$ of $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{g})$. The subalgebra $\mathfrak{k}$ is called a *compactly imbedded subalgebra of* $\mathfrak{g}$ if K^* is compact. The Lie algebra $\mathfrak{g}$ is said to be *compact* if it is compactly imbedded in itself or equivalently if Int $(\mathfrak{g})$ is compact.

It should be observed that the topology of K^* might *a priori* differ from the relative topology of the group Int $(\mathfrak{g})$ which again might differ from the relative topology of $GL(\mathfrak{g})$. The next proposition clarifies this situation.

Proposition 5.4. *Let $\tilde{K}$ denote the abstract group K^* with the relative topology of $GL(\mathfrak{g})$. Then K^* is compact if and only if $\tilde{K}$ is compact.*

The identity mapping of K^* into $GL(\mathfrak{g})$ is analytic, in particular, continuous. Thus $\tilde{K}$ is compact if K^* is compact. On the other hand, if $\tilde{K}$ is compact, then it is closed in $GL(\mathfrak{g})$; by Theorem 2.10, K^* and $\tilde{K}$ are homeomorphic.

Remark. Suppose G is any connected Lie group with Lie algebra $\mathfrak{g}$. Let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Then the group K^* above coincides with $\operatorname{Ad}_G(K)$; in fact, both groups are generated by $\operatorname{Ad}_G(\exp X)$, $X \in \mathfrak{k}$.

§ 6. Semisimple Lie Groups

Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0. Denoting by Tr the trace of a vector space endomorphism we consider the bilinear form $B(X, Y) = \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ on $\mathfrak{g} \times \mathfrak{g}$. The form B is called the *Killing form* of $\mathfrak{g}$. It is clearly symmetric.

For a subspace $\mathfrak{a} \subset \mathfrak{g}$ let $\mathfrak{a}^{\perp} = \{X : B(X, \mathfrak{a}) = 0\}$. The map $X \to X^*$ of $\mathfrak{g} \to \widehat{\mathfrak{g}}$ (dual of $\mathfrak{g}$) given by $X^*(Y) = B(X, Y)$ has kernel $\mathfrak{g}^{\perp}$. Let $V \subset \widehat{\mathfrak{g}}$ and $V' = \{X \in \mathfrak{g} : V(X) = 0\}$. By linear algebra, $\dim V + \dim V' = \dim \mathfrak{g}$. If $V = \mathfrak{a}^*$, the image of $\mathfrak{a}$ under $X \to X^*$, then $V' = \mathfrak{a}^{\perp}$ so

$$\dim \mathfrak{a}^* + \dim \mathfrak{a}^{\perp} = \dim \mathfrak{g}\,.$$

But then map $X \to X^*$ from $\mathfrak{a}$ to $\mathfrak{a}^*$ has kernel $\mathfrak{a} \cap \mathfrak{g}^{\perp}$ so

$$\dim \mathfrak{a} = \dim \mathfrak{a}^* + \dim(\mathfrak{a} \cap \mathfrak{g}^{\perp})\,.$$

Eliminating $\mathfrak{a}^*$ we get

$$\dim \mathfrak{a} + \dim \mathfrak{a}^{\perp} = \dim \mathfrak{g} + \dim(\mathfrak{a} \cap \mathfrak{g}^{\perp}) \tag{1}$$

If σ is an automorphism of $\mathfrak{g}$, then $\mathrm{ad}(\sigma X) = \sigma \circ \mathrm{ad}\, X \circ \sigma^{-1}$ so by $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, we have

$$B(\sigma X, \sigma Y) = B(X, Y), \qquad \sigma \in \mathrm{Aut}(\mathfrak{g}),$$
$$B(X, [Y, Z]) = B(Y, [Z, X]) = B(Z, [X, Y]), \qquad X, Y, Z \in \mathfrak{g}. \tag{2}$$

Suppose $\mathfrak{a}$ is an ideal in $\mathfrak{g}$. Then it is easily verified that the Killing form of $\mathfrak{a}$ coincides with the restriction of B to $\mathfrak{a} \times \mathfrak{a}$.

Definition. A Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is called *semisimple* if the Killing B of $\mathfrak{g}$ is nondegenerate. We shall call a Lie algebra $\mathfrak{g} \neq \{0\}$ *simple*† if it is semisimple and has no ideals except $\{0\}$ and $\mathfrak{g}$. A Lie group is called semisimple (simple) if its Lie algebra is semisimple (simple).

Proposition 6.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{a}$ an ideal in $\mathfrak{g}$. Let $\mathfrak{a}^{\perp}$ denote the set of elements $X \in \mathfrak{g}$ which are orthogonal to $\mathfrak{a}$ with respect to B. Then $\mathfrak{a}$ is semisimple, $\mathfrak{a}^{\perp}$ is an ideal and*

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{a}^{\perp} \quad (\textit{direct sum}).$$

Proof. The fact that $\mathfrak{a}^{\perp}$ is an ideal is obvious from (2). Since B is nondegenerate, (1) implies $\dim \mathfrak{a} + \dim \mathfrak{a}^{\perp} = \dim \mathfrak{g}$. If $Z \in \mathfrak{g}$ and $X, Y \in \mathfrak{a} \cap \mathfrak{a}^{\perp}$, we have $B(Z, [X, Y]) = B([Z, X], Y) = 0$ so $[X, Y] = 0$. Hence $\mathfrak{a} \cap \mathfrak{a}^{\perp}$ is an abelian ideal in $\mathfrak{g}$. Let $\mathfrak{b}$ be any subspace of $\mathfrak{g}$ complementary to $\mathfrak{a} \cap \mathfrak{a}^{\perp}$. If $Z \in \mathfrak{g}$ and $T \in \mathfrak{a} \cap \mathfrak{a}^{\perp}$, then the endomorphism $\mathrm{ad}\, T\, \mathrm{ad}\, Z$ maps $\mathfrak{a} \cap \mathfrak{a}^{\perp}$ into $\{0\}$, and $\mathfrak{b}$ into $\mathfrak{a} \cap \mathfrak{a}^{\perp}$. In particular, $\mathrm{Tr}(\mathrm{ad}\, T\, \mathrm{ad}\, Z) = 0$. It follows that $\mathfrak{a} \cap \mathfrak{a}^{\perp} = \{0\}$ and we get the direct decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{a}^{\perp}$. Since the Killing form of $\mathfrak{a}$ is the restriction of B to $\mathfrak{a} \times \mathfrak{a}$, the semisimplicity of $\mathfrak{a}$ is obvious.

† This definition of a simple Lie algebra is convenient for our purposes but is formally different from the usual one: A Lie algebra $\mathfrak{g}$ is simple if it is nonabelian and has no ideals except $\{0\}$ and $\mathfrak{g}$. However, the two definitions are equivalent.

Corollary 6.2. *A semisimple Lie algebra has center* $\{0\}$.

Corollary 6.3. *A semisimple Lie algebra* $\mathfrak{g}$ *is the direct sum*

$$\mathfrak{g} = \mathfrak{g}_1 + \dots + \mathfrak{g}_r,$$

where $\mathfrak{g}_i$ $(1 \leqslant i \leqslant r)$ *are all the simple ideals in* $\mathfrak{g}$. *Each ideal* $\mathfrak{a}$ *of* $\mathfrak{g}$ *is the direct sum of certain* $\mathfrak{g}_i$.

In fact, Prop. 6.1 implies that $\mathfrak{g}$ can be written as a direct sum of simple ideals $\mathfrak{g}_i$ $(1 \leqslant i \leqslant s)$ such that $\mathfrak{a}$ is the direct sum of certain of these $\mathfrak{g}_i$. If $\mathfrak{b}$ were a simple ideal which does not occur among the ideals $\mathfrak{g}_i$ $(1 \leqslant i \leqslant s)$, then $[\mathfrak{g}_i, \mathfrak{b}] \subset \mathfrak{g}_i \cap \mathfrak{b} = \{0\}$ for $1 \leqslant i \leqslant s$. This contradicts Cor. 6.2.

Proposition 6.4. *If* $\mathfrak{g}$ *is semisimple, then* $ad(\mathfrak{g}) = \partial(\mathfrak{g})$, *that is, every derivation is an inner derivation.*

Proof. The algebra $\mathrm{ad}(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$, hence semisimple. If D is a derivation of $\mathfrak{g}$ then $\mathrm{ad}(DX) = [D, \mathrm{ad}\, X]$ for $X \in \mathfrak{g}$, hence $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\partial(\mathfrak{g})$. Its orthogonal complement, say $\mathfrak{a}$, is also an ideal in $\partial(\mathfrak{g})$. Then $\mathfrak{a} \cap \mathrm{ad}(\mathfrak{g})$ is orthogonal to $\mathrm{ad}(\mathfrak{g})$ also with respect to the Killing form of $\mathrm{ad}(\mathfrak{g})$, hence $\mathfrak{a} \cap \mathrm{ad}(\mathfrak{g}) = \{0\}$. Consequently $D \in \mathfrak{a}$ implies $[D, \mathrm{ad}\, X] \in \mathfrak{a} \cap \mathrm{ad}(\mathfrak{g}) = \{0\}$. Thus $\mathrm{ad}(DX) = 0$ for each $X \in \mathfrak{g}$, hence $D = 0$. Thus, $\mathfrak{a} = \{0\}$ so, by (1), $\mathrm{ad}(\mathfrak{g}) = \partial(\mathfrak{g})$.

Corollary 6.5. *For a semisimple Lie algebra* $\mathfrak{g}$ *over* $\boldsymbol{R}$, *the adjoint group* $Int(\mathfrak{g})$ *is the identity component of* $Aut(\mathfrak{g})$. *In particular,* $Int(\mathfrak{g})$ *is a closed topological subgroup of* $Aut(\mathfrak{g})$.

Remark. If $\mathfrak{g}$ is not semisimple, the group $\mathrm{Int}(\mathfrak{g})$ is not necessarily closed in $\mathrm{Aut}(\mathfrak{g})$ (see Exercise D.3 for this chapter).

Proposition 6.6.

(i) *Let* $\mathfrak{g}$ *be a semisimple Lie algebra over* $\boldsymbol{R}$. *Then* $\mathfrak{g}$ *is compact if and only if the Killing form of* $\mathfrak{g}$ *is strictly negative definite.*

(ii) *Every compact Lie algebra* $\mathfrak{g}$ *is the direct sum* $\mathfrak{g} = \mathfrak{z} + [\mathfrak{g}, \mathfrak{g}]$ *where* $\mathfrak{z}$ *is the center of* $\mathfrak{g}$ *and the ideal* $[\mathfrak{g}, \mathfrak{g}]$ *is semisimple and compact.*

Proof. Suppose $\mathfrak{g}$ is a Lie algebra over $\boldsymbol{R}$ whose Killing form is strictly negative definite. Let $O(B)$ denote the group of all linear transformations of $\mathfrak{g}$ which leave B invariant. Then $O(B)$ is compact in the relative topology of $GL(\mathfrak{g})$. We have $\mathrm{Aut}(\mathfrak{g}) \subset O(B)$, so by Cor. 6.5, $\mathrm{Int}(\mathfrak{g})$ is compact.

Suppose now $\mathfrak{g}$ is an arbitrary compact Lie algebra. The Lie subgroup $\mathrm{Int}(\mathfrak{g})$ of $GL(\mathfrak{g})$ is compact; hence it carries the relative topology of

$GL(\mathfrak{g})$. There exists a strictly positive definite quadratic form Q on $\mathfrak{g}$ invariant under the action of the compact linear group $\operatorname{Int}(\mathfrak{g})$. There exists a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ such that $Q(X) = \sum_{i=1}^n x_i^2$ if $X = \sum_{i=1}^n x_i X_i$. By means of this basis, each $\sigma \in \operatorname{Int}(\mathfrak{g})$ is represented by an orthogonal matrix and each $\operatorname{ad} X$, $(X \in \mathfrak{g})$, by a skew symmetric matrix, say $(a_{ij}(X))$. Now the center $\mathfrak{z}$ of $\mathfrak{g}$ is invariant under $\operatorname{Int}(\mathfrak{g})$, that is $\sigma \cdot \mathfrak{z} \subset \mathfrak{z}$ for each $\sigma \in \operatorname{Int}(\mathfrak{g})$. The orthogonal complement $\mathfrak{g}'$ of $\mathfrak{z}$ in $\mathfrak{g}$ with respect to Q is also invariant under $\operatorname{Int}(\mathfrak{g})$ and under $\operatorname{ad}(\mathfrak{g})$. Hence $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$. This being so, the Killing form B' of $\mathfrak{g}'$ is the restriction to $\mathfrak{g}' \times \mathfrak{g}'$ of the Killing form B of $\mathfrak{g}$. Now, if $X \in \mathfrak{g}$

$$B(X, X) = \operatorname{Tr}(\operatorname{ad} X \operatorname{ad} X) = \sum_{i,j} a_{ij}(X)\, a_{ji}(X) = -\sum_{i,j} a_{ij}(X)^2 \leqslant 0.$$

The equality sign holds if and only if $\operatorname{ad} X = 0$, that is, if and only if $X \in \mathfrak{z}$. This proves that $\mathfrak{g}'$ is semisimple and compact. The decomposition in Cor. 6.3 shows that $[\mathfrak{g}', \mathfrak{g}'] = \mathfrak{g}'$. Hence $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and the proposition is proved.

Corollary 6.7. *A Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ is compact if and only if there exists a compact Lie group G with Lie algebra isomorphic to $\mathfrak{g}$.*

For this corollary one just has to remark that every abelian Lie algebra is isomorphic to the Lie algebra of a torus $\boldsymbol{S}^1 \times \ldots \times \boldsymbol{S}^1$.

Proposition 6.8. *Let $\mathfrak{g}$ be a Lie algebra over $\boldsymbol{R}$ and let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. Suppose $\mathfrak{k}$ is a compactly imbedded subalgebra of $\mathfrak{g}$. If $\mathfrak{k} \cap \mathfrak{z} = \{0\}$ then the Killing form of $\mathfrak{g}$ is strictly negative definite on $\mathfrak{k}$.*

Proof. Let B denote the Killing form of $\mathfrak{g}$, and let K denote the analytic subgroup of the adjoint group $\operatorname{Int}(\mathfrak{g})$ with Lie algebra $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{k})$. Owing to our assumptions, K is a compact Lie subgroup of $\boldsymbol{GL}(\mathfrak{g})$; hence it carries the relative topology of $\boldsymbol{GL}(\mathfrak{g})$. There exists a strictly positive definite quadratic form Q on $\mathfrak{g}$ invariant under K. There exists a basis of $\mathfrak{g}$ such that each endomorphism $\operatorname{ad}_{\mathfrak{g}}(T)$ $(T \in \mathfrak{k})$ is expressed by means of a skew symmetric matrix, say $(a_{ij}(T))$. Then

$$B(T, T) = \sum_{i,j} a_{ij}(T)\, a_{ji}(T) = -\sum_{i,j} a_{ij}(T)^2 \leqslant 0,$$

and equality sign holds only if $T \in \mathfrak{z} \cap \mathfrak{k} = \{0\}$.

Theorem 6.9. *Let G be a compact, connected semisimple Lie group. Then the universal covering group G^* of G is compact.*

There are several proofs of this result in the literature but Samelson's geometric proof, which we give below, is rarely to be seen. It relies on a few basic results in Riemannian geometry. These results will be familiar to many readers but in any case their proofs can be found in Chapter I of the text. These results are the following.

(i) If (M, π) is a covering manifold of a complete Riemannian manifold N with Riemannian structure g then M with the Riemannian structure $\pi^* g$ is also complete. (Proposition 10.6, Ch. I.)

(ii) Let o be a point in a complete, noncompact Riemannian manifold M. Then M contains a ray with initial point o (i.e., a geodesic which realizes the shortest distance between any two of its points). (Proposition 10.7, Ch. I.)

(iii) If in a Riemannian manifold a piecewise differentiable curve γ joining p to q has length $L(\gamma) = d(p, q)$ then it is geodesic.

(iv) In a complete Riemannian manifold any two points can be joined by a geodesic. (Theorem 10.4, Ch. I.)

The universal covering group will be covered in the next section.

Proof. Let $\mathfrak{g}$ denote the Lie algebra of G (and G^*), and let B be the Killing form of $\mathfrak{g}$. There exists unique left invariant Riemannian structures Q and Q^* on G and G^*, respectively, such that $Q_e = Q^*_{e^*} = -B$. Here e and e^* denote the identity elements in G and G^*, respectively. Since

$$B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = B(X, Y), \qquad X, Y \in \mathfrak{g}, g \in G,$$

it follows that Q and Q^* are also invariant under right translations on G and G^*. Let π denote the covering mapping of G^* onto G. Then $Q^* = \pi^* Q$ and since G is complete, the covering manifold G^* is also complete (by (i)). Because of Exercise A6, Ch. II the geodesics in G^* through e^* are the one-parameter subgroups.

Suppose now the theorem were false for G. Then, due to Prop. 10.7, Chapter I, G^* contains a ray emanating from e^*. Let γ be the one-parameter subgroup containing this ray. Then γ is a "straight line" in G^*, that is, it realizes the shortest distance in G^* between any two of its points. In fact, any pair of points on γ can be moved by a left translation on a pair of points on the ray. We parametrize γ by arc length t measured from the point $e^* = \gamma(0)$. The set $\pi(\gamma)$ is a one-parameter subgroup of G; its closure in G is a compact, abelian, connected subgroup, hence a torus. By the classical theorem of Kronecker, there exists a sequence $(t_n) \subset R$ such that $t_n \to \infty$ and $\pi(\gamma(t_n)) \to e$. We can assume that all $\pi(\gamma(t_n))$ lie in a minimizing convex normal ball $B_r(e)$ and that each component C of $\pi^{-1}(B_r(e))$ is diffeomorphic to $B_r(e)$ under π. Then the mapping $\pi : C \to B_r(e)$ is distance-preserving; hence there exists an element $z_n \in G^*$ such that

$$\pi(z_n) = e, \tag{3}$$
$$d(z_n, \gamma(t_n)) = d(e, \pi(\gamma(t_n))).$$

Here d denotes the distance in G as well as in G^*. Since (G^*, π) is a covering group of G, the kernel of π is contained in the center Z of G^*. Hence by (3), we have $z_n \in Z$. We intend to show $\gamma \subset Z$.

Now for a given element $a \in G^*$, consider the one-parameter subgroup $\delta : t \to a\gamma(t)a^{-1}$ $(t \in R)$. Since left and right translations on G^* are isometries, δ is a "straight line" and $|t|$ is the arc parameter measured from e^*. Since $z_n \in Z$ we have $d(\delta(t_n), z_n) = d(\gamma(t_n), z_n)$ and this shows that

$$\lim_{n\to\infty} d(\gamma(t_n), \delta(t_n)) = 0. \tag{4}$$

Suppose now $\delta(t) \neq \gamma(t)$ for some $t \neq 0$. Then the angle between the vectors $\dot{\gamma}(0)$ and $\dot{\delta}(0)$ is different from 0 (possibly 180°). In any case, we have from (iii) above

$$d(\gamma(-1), \delta(+1)) < d(e^*, \gamma(-1)) + d(e^*, \delta(1)) = 2.$$

From (4) we can determine an integer N such that

$$t_N > 1, \quad d(\gamma(t_N), \delta(t_N)) < 2 - d(\gamma(-1), \delta(+1)).$$

We consider now the following broken geodesic ζ: from $\gamma(-1)$ to $\delta(+1)$ along a shortest geodesic, from $\delta(+1)$ to $\delta(t_N)$ on δ, from $\delta(t_N)$ to $\gamma(t_N)$ along a shortest geodesic. The curve ζ joins $\gamma(-1)$ to $\gamma(t_N)$ and has length

$$d(\gamma(-1), \delta(+1)) + (t_N - 1) + d(\delta(t_N), \gamma(t_N)),$$

which is strictly smaller than $t_N + 1 = d(\gamma(-1), \gamma(t_N))$. This contradicts the property of γ being a straight line.

It follows that $\delta(t) = \gamma(t)$ for all $t \in R$. Since $a \in G^*$ was arbitrary it follows that $\gamma \subset Z$. But then $\mathfrak{z}$, the Lie algebra of Z, is $\neq \{0\}$, and this contradicts the semisimplicity of $\mathfrak{g}$.

Proposition 6.10. *Let G be a connected Lie group with compact Lie algebra $\mathfrak{g}$. Then the mapping* $\exp : \mathfrak{g} \to G$ *is surjective.*

The proof is contained in the first part of the proof of Theorem 6.9. In fact, $\mathrm{Ad}(G)$ being compact, there exists a strictly positive definite quadratic form on $\mathfrak{g}$ invariant under $\mathrm{Ad}(G)$. In the corresponding left and right invariant Riemannian metric on G the geodesics through e are the one-parameter subgroups. Thus G is complete and the result follows from (iv) above.

7 The Universal Covering Group

We shall now sketch the construction of the universal covering group, relying on Pontrjagin's *Topological Groups* for proofs of some of the results.

Let M and N be connected and locally connected spaces and $\pi : M \rightarrow N$ a continuous mapping. The pair (M, π) is called a ***covering space*** of N if each point $n \in N$ has an open neighborhood U such that each component of $\pi^{-1}(U)$ is homeomorphic to U under π.

Suppose N is a differentiable manifold and that (M, π) is a covering space of N. Then there is a unique differentiable structure on M such that the mapping π is regular. If M is given this differentiable structure, we say that (M, π) is a ***covering manifold*** of N.

We shall require the following standard theorem from the theory of covering spaces. We state it only for manifolds although it holds under suitable local connectedness hypotheses.

Let (M, π) be a covering manifold of N and let $\Gamma : [a, b] \rightarrow N$ be a path in N. If m is any point in M such that $\pi(m) = \Gamma(a)$, there exists a unique path $\Gamma^* : [a, b] \rightarrow M$ such that $\Gamma^*(a) = m$ and $\pi \circ \Gamma^* = \Gamma$.

The path Γ^* is called the *lift* of Γ through m.

Proposition *Let N be a Riemannian manifold with a Riemannian structure g. Let (M, π) be a covering manifold of N. Then π^*g is a Riemannian structure on M. Moreover, M is complete if and only if N is complete.*

Proof. The mapping π is regular, so obviously π^*g is a Riemannian structure on M. If γ is a curve segment in M, then $\pi \circ \gamma$ is a curve segment in N. Using the characterization of geodesics by means of differential equations (3), (§ 5), it is clear that γ is a geodesic if and only if $\pi \circ \gamma$ is a geodesic. But completeness is equivalent to the infiniteness of each maximal geodesic. The proposition follows immediately.

Consider a continuous path $\ell : t \to f(t)$ $(0 \le t \le 1)$ and its inverse ℓ^{-1} : $t \to f(1-t)$. If $k : t \to g(t)$ $(0 \le t \le 1)$ is another path and $f(1) = g(0)$ then $k\ell : t \to h(t)$ $(0 \le t \le 1)$ is the path defined by

$$h(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

ℓ is *closed* if $f(0) = f(1)$. We say k is *homotopic* to ℓ, denoted $k \sim \ell$ if there is a continuous function on $0 \le t \le 1$, $0 \le s \le 1$ such that $\varphi(0,t) = f(t)$, $\varphi(1,t) = g(t)$. Fix $p \in M$ and let P denote the set of closed paths starting at p. The homotopy classes in P form a group, the *fundamental group* of M. Up to isomorphism this is independent of the point p. M is *simply connected* if the fundamental group is e.

The *universal covering space* $\widetilde{N}$ of N above is constructed as follows. Let $\widetilde{N}$ denote the set of equivalence classes of paths, starting at a point $p \in N$. Then we have a map $\varphi : \widetilde{N} \to N$ mapping each path into its endpoint. Given a neighborhood of a point in N let U^* denote the set of equivalence classes of paths ending in U. These U^* topologize the set $\widetilde{N}$ and the map $\varphi : \widetilde{N} \to N$ is a covering, the universal covering space of N. If $(\widetilde{N}, \varphi)$ and $(\widetilde{N}_1, \varphi_1)$ are two universal covering spaces of N then there is a homeomorphism $\tau : \widetilde{N} \to \widetilde{N}_1$ such that $\varphi_1 \circ \tau = \varphi$. $\widetilde{N}$ is simply connected.

Suppose now G is a connected topological group. A *covering group* of G is a covering space $(\widetilde{G}, \pi)$ of G such that $\widetilde{G}$ is a topological group and $\pi : \widetilde{G} \to G$ is a homomorphism. Suppose $(\widetilde{G}, \pi)$ is a universal covering space of G. We make $\widetilde{G}$ into a group as follows. Let $A, B \in \widetilde{G}$, let k be a path in the class A, ℓ a path in class B, both starting at e. Let $\pi(A) = a$, $\pi(B) = b$, a and b being the end points of k and ℓ, respectively. If $f(t)$ is the path of ℓ, $af(t)$ is a new path, denoted $a\ell$. Then $\ell \sim \ell'$ implies $a\ell \sim a\ell'$. Since k ends at a, k and $a\ell$ can be composed. Let C denote the class of the composite path $k(a\ell)$. This turns $\widetilde{G}$ into a group and $(\widetilde{G}, \pi)$ is a covering group of G.

If G is a Lie group, and we give $\widetilde{G}$ the analytic structure such that π is regular then it is not hard to show that $\widetilde{G}$ is a Lie group. The kernel D of π is a normal subgroup which is discrete. In fact, if U is a neighborhood of the identity in $\widetilde{G}$ on which π is injective we have $D \cap U = (e)$ so e is open in D. We also have

Lemma 7.1. *A discrete normal subgroup D of a connected topological group G is contained in the center.*

In fact let $d \in D$ and N a neighborhood of d such that $N \cap D = (d)$ and let V be a neighborhood of e in G such that $VdV^{-1} \subset N$. Since D is normal, $\sigma \in V$ implies $\sigma d \sigma^{-1} \subset N \cap D = d$. Since G is connected, V generates G so $gdg^{-1} = d$ for all $g \in G$.

Theorem 7.2. *Let G be a connected, locally arcwise connected topological group. Let $H \subset G$ be a closed subgroup and H_0 the identity component of H. Then*

(i) G/H is connected and locally arcwise connected.

(ii) The natural map $G/H_0 \to G/H$ is a covering.

(iii) If G/H is simply connected then $H = H_0$.

(iv) If H is discrete, $G \to G/H$ is a covering.

Proof: (i) The natural map $\pi : G \to G/H$ is continuous so G/H is connected. If a is in an open subset $V \subset G/H$ take $\tilde{a} \in \pi^{-1}(a)$ and a connected arcwise connected neighborhood W of $\tilde{a}$ contained in $\pi^{-1}(V)$. Then $\pi(W)$ is an arcwise connected neighborhood of a contained in V.

For (ii) let $\pi_0 : G \to G/H_0$, $\pi : G \to G/H$ and $\sigma : G/H_0 \to G/H$ be the natural maps. If $U \subset G/H$ is open then $\sigma^{-1}(U) = \pi_0(\pi^{-1}(U))$ is open so σ is continuous. Also if $V \subset G/H_0$ is open, $\sigma(V) = \pi(\pi_0^{-1}(V))$ is open so σ is a continuous open mapping. H is locally connected so H_0 is open in H. Thus there exists an open subset $V \subset G$ such that $V \cap H = H_0$. Choose a connected neighborhood U of e in G such that $U^{-1}U \subset V$ and $U^{-1}U \cap H \subset H_0$. Then UH is a neighborhood of the origin in G/H.

Consider the inverse image

$$\sigma^{-1}(UH) = \{gH_0 : \sigma(g) \in UH\} = \{UhH_0 \in G/H_0 : h \in H\}$$

the latter equality holding since $gH \in UH$ implies $g = uh$. Each UhH_0 is open in G/H_0 and is connected and their union is $\sigma^{-1}(UH)$. If

$$Uh_1H_0 \cap Uh_2H_0 \neq \emptyset \tag{1}$$

then since H_0 is normal in H,

$$UH_0h_1 \cap UH_0h_2 \neq \emptyset,$$

and hence $U^{-1}UH_0 \cap H_0h_2h_1^{-1} \neq \emptyset$. Again since H_0 is normal in H,

$$U^{-1}U \cap H_0h_2^{-1}h_1 \neq \emptyset.$$

Thus $U^{-1}U$ contains an element $h \in H$ so since $U^{-1}U \cap H \subset H_0$, $h \in H_0$ so $h_2^{-1}h_1 \in H_0$. Thus (1) implies

$$Uh_1H_0 = Uh_2H_0\,. \tag{2}$$

We claim now that for each $h \in H$ the map $\sigma : UhH_0 \to UH$ is a bijection. In fact if $\sigma(u_1hH_0) = \sigma(u_2hH_0)$ then $u_1h = u_2hh^*$ for some $h^* \in H$ so $u_2^{-1}u_1 \in H$ so by $U^{-1}U \cap H \subset H_0$, $u_1 = u_2h_0$ for some $h_0 \in H_0$, whence the injectivity of σ. The surjectivity is obvious. The sets UhH_0 being the components of $\sigma^{-1}(UH)$, UH is evenly covered. By translation we see that σ is a covering. Now (iii) and (iv) follow from (ii).

Theorem 7.3. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Assume G simply connected. If $\sigma : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism there exists an analytic homomorphism $\mathcal{S} : G \to H$ such that $d\mathcal{S} = \sigma$.*

Proof:

Consider the product group $G \times H$ and the two projections $p_1 : G \times H \to G$, $p_2 : G \times H \to H$ given by $p_1(g,h) = g$, $p_2(g,h) = h$ which have differentials $dp_1(X,Y) = X$, $dp_2(X,Y) = Y$. Let $\mathfrak{k}$ denote the graph $\{(X, \sigma X) : X \in \mathfrak{g}\}$ and K the corresponding analytic subgroup of $G \times H$. The restriction $\varphi = p_1|_K$ has differential $d\varphi = dp_1|_{\mathfrak{k}}$ which is the isomorphism $(X, \sigma X) \to X$ of $\mathfrak{k}$ onto $\mathfrak{g}$. Thus $\varphi : K \to G$ is a surjective homomorphism with a discrete kernel, hence by Theorem 7.2 a covering. But G is simply connected so φ is an isomorphism. The homomorphism $p_2 \circ \varphi^{-1}$ of G into H has the differential σ.

The automorphism group. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, $(\widetilde{G}, \pi)$ its universal covering group so $G = \widetilde{G}/D$ where D is a discrete central subgroup. The group $\mathrm{Aut}\,(\widetilde{G})$ of automorphisms of $\widetilde{G}$ is by Theorem 7.3 identified with $\mathrm{Aut}\,(\mathfrak{g})$ and is thus a Lie group. Then $\mathrm{Aut}\,(G)$ is identified with the closed subgroup preserving D and is thus also a Lie group. In fact, if $\sigma \in \mathrm{Aut}\,(G)$ there exists an automorphism θ of $\widetilde{G}$ such that $d\theta = d\pi^{-1} \circ d\sigma \circ d\pi$. Then $\pi\theta = \sigma\pi$ so θ maps D into itself.