18.905 Problem Set 2

Due Wednesday, September 20 in class

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1. A *simplicial complex* consists of a pair (V, F), where V is a set of vertices and F (the set of faces) is a collection of finite subsets of V satisfying the following properties.

   - We have $\{v\} \in F$ for all $v \in V$.
   - If $S \subset T$ and $T \in F$, then $S \in F$.

   (Course 6 people might know this as a "hereditary hypergraph".) From a simplicial complex (V, F), we form a space X by starting with the vertices V and, for every $S \in F$ of size $n+1$, we glue in a unique n-simplex whose vertices are the elements of S. The faces of this simplex correspond to the subsets of S.

   More precisely, let W be the vector space with basis $\{e_v | v \in V\}$, and let

   $$X = \bigcup_{S \in F} \left\{ \sum_{v \in S} t_v v \middle| 0 \le t_v \le 1, \sum t_v = 1 \right\}.$$

   Suppose that we have chosen a partial order on V such that for any $S \in F$, the elements of S are totally ordered. Use this to give a Δ-complex structure on X. (You may assume that for any $v_0, \ldots, v_n$ in a vector space W, there is a unique affine transformation $f : \Delta^n \to W$ such that f takes the i'th vertex of Δ^n to v_i.)

   Update. It has been pointed out to me that I need to be explicit about what the topology on the vector space W is; it's not the metric topology or the product topology. The topology on W is a limit topology: A subspace $A \subset W$ is closed if and only if $A \cap U$ is closed for any finite-dimensional subspace U of W.

2. Hatcher, exercise 8 on page 131.

3. In class, we defined face maps $d_n^i : \Delta^{n-1} \to \Delta^n$ for $0 \le i \le n$, and subdivision maps $s_n^j : \Delta^{n+1} \to \Delta^n \times [0,1]$ for $1 \le j \le n+1$. These satisfy the following relations.

$$\begin{aligned}
s_n^j \circ d_{n+1}^i &= \begin{cases} (d_n^{i-1}, id) \circ s_{n-1}^j & \text{if } j < i \\ (d_n^i, id) \circ s_{n-1}^{j-1} & \text{if } j > i+1 \end{cases} \\
s_n^1 \circ d_{n+1}^0 &= (id, 1) \\
s_n^{n+1} \circ d_{n+1}^{n+1} &= (id, 0) \\
s_n^i \circ d_{n+1}^i &= s_n^{i+1} \circ d_{n+1}^i
\end{aligned}$$

If $H : X \times [0,1] \to Y$ is a homotopy between the maps f and g, we then defined a homotopy operator $h : C_n(X) \to C_{n+1}(Y)$ by

$$h(\sum n_\sigma[\sigma]) = \sum n_\sigma \sum_{j=1}^{n+1} (-1)^j [H \circ (\sigma, id) \circ s_n^j],$$

where the map $[H \circ (\sigma, id) \circ s_n^j]$ is the composite map $\Delta^{n+1} \to Y$.

Use the given relations to show that for any $\sigma : \Delta^n \to X$, we have

$$\partial h(\sigma) = f_*(\sigma) - g_*(\sigma) - h(\partial\sigma)$$

in $C_{n+1}(Y)$.

4. Suppose $f : A \to B$ and $g : B \to C$ are homomorphisms of abelian groups. Show that there is an exact sequence

$$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0.$$