18.905 Problem Set 3

Due Wednesday, September 27 in class

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1. Suppose that C_* and D_* are chain complexes, and f_* and g_* are maps of chain complexes from C_* to D_*. Recall that a *chain homotopy* from f to g is a collection of maps $h_n : C_n \to D_{n+1}$ such that for all $x \in C_n$,

$$\partial h_n x + h_{n-1} \partial x = f_n x - g_n x.$$

   If a chain homotopy exists, then f_* and g_* induce the same map on homology.

   Find an example of two maps of chain complexes which give the same map on homology, but for which there is no chain homotopy.

2. Suppose $\sigma : [0,1] \to X$ is a 1-simplex. Define $\overline{\sigma}(t) = \sigma(1-t)$, the same simplex with its direction reversed. Find an element $u \in C_2(X)$ such that $\partial u = \sigma + \overline{\sigma}$ (so $\overline{\sigma}$ can always be exchanged for $-\sigma$ in homology).

3. Suppose $A \subset B \subset C$ are spaces. Show that there is a long exact sequence of homology groups as follows.

$$\cdots \to H_{n+1}(C,B) \to H_n(B,A) \to H_n(C,A) \to H_n(C,B) \to H_{n-1}(B,A) \to \cdots$$

4. Fix a space Y. For a space X with a subspace A, define

$$H_n^Y(X,A) = H_n(X \times Y, A \times Y).$$

   Show that H_n^Y satisfies all of the Eilenberg-Steenrod axioms except for the dimension axiom.

   Note: This means that you need to show:

   - A map $f : X \to Z$ such that $f(A) \subset B$ induces a map $f_* : H_n^Y(X,A) \to H_n^Y(Z,B)$, and $(g \circ f)_* = g_* \circ f_*$.
   - If f and g are two maps $X \to Z$ such that $f(A) \subset B$ and $g(A) \subset B$, and there is a homotopy H from f to g such that $H(a,t) \in B$ for all $a \in A, t \in [0,1]$, then $f_* = g_*$.
   - If $V \subset A$ is a subspace such that the closure of V is contained in the interior of A, then the map $H_n^Y(X \setminus V, A \setminus V) \to H_n^Y(X,A)$ is an isomorphism.
   - There are boundary maps $\partial : H_n^Y(X,A) \to H_{n-1}^Y(A)$ such that the sequence of maps

$$\cdots \to H_{n+1}^Y(X,A) \to H_n^Y(A) \to H_n^Y(X) \to H_n^Y(X,A) \to H_{n-1}^Y(A) \to \cdots$$

     is exact. Additionally, if $f : X \to Z$ is a map with $f(A) \subset B$, then $\partial \circ f_* = f_* \circ \partial$.
   - If X is a disjoint union of disconnected subspaces X_α, then $H_n^Y(X) = \oplus_\alpha H_n^Y(X_\alpha)$.